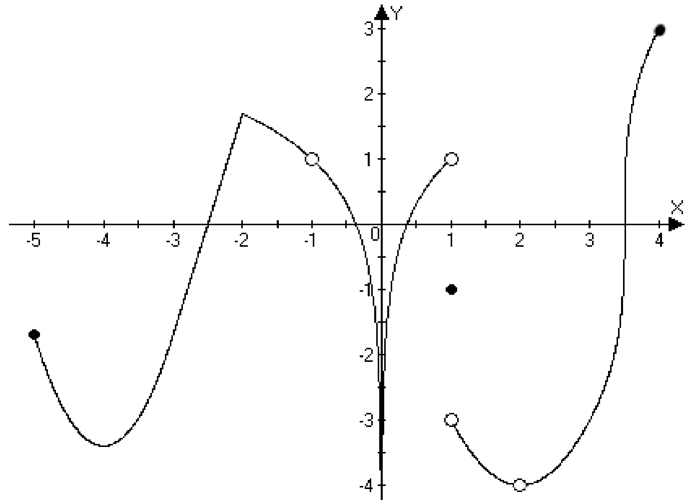
Math 1A – Chapter 2 Test – Spring '08 Name:________________________________
Show your work for credit. Write all responses on separate paper. Do not abuse a calculator.

1. The graph of $y = f(x)$ is given. Assume that $x = 0$ is a vertical asymptote



   a. Estimate $\lim_{x\to -1} f(x)$ if it exists, or explain why it doesn't exist.
   b. Estimate $\lim_{x\to 1^+} f(x)$ if it exists, or explain why it doesn't exist.
   c. Estimate $\lim_{x\to 1} f(x)$ if it exists, or explain why it doesn't exist.
   d. For what value(s) of a in $[-5,4]$ does $\lim_{x\to a} f(x)$ not exist?
   e. For what value(s) of a in $[-5,4]$does $\lim_{x\to a} f(x)$ exist, while f is discontinuous?
   f. Where does the derivative function $f'(x)$ have a jump discontinuity?

2. Find the limit. Explain your answers.

   a. $\lim_{x\to 3}\frac{(x-1)^2-1}{x^2-9}$ b. $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$ c. $\lim_{x\to\infty}\frac{\sin x}{1+\ln x}$ d. $\lim_{x\to\infty} e^{\sin x}$

3. Use the intermediate value theorem to prove that $\frac{18}{\pi^2}x^2 = \tan x$ has a solution in $\left(\frac{\pi}{6},\frac{\pi}{4}\right)$.

4. If the tangent to $y = f(x)$ at (0,3) passes through the point (2,0), find $f'(0)$.

5. Find the derivative function for $f(x) = \frac{1}{x+1}$ using the definition of the derivative.

6. Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \frac{1}{\sqrt{x^3+1}}$ for all x.

   a. Find an equation for the tangent line at (2,–4).

   b. Use the tangent line approximation to estimate $g(1.95)$ and $g(2.05)$.

   c. Are your estimates in part (a) too large or too small? Explain.

7. Is there a number a such that $\lim_{x\to 2}\frac{x-a}{x^2+x-6}$ exists? If not, why not? If so, find the value of a and the value of the limit.

8. Consider $\lim_{x\to 0}\arctan\left(x+e^x\right)$.

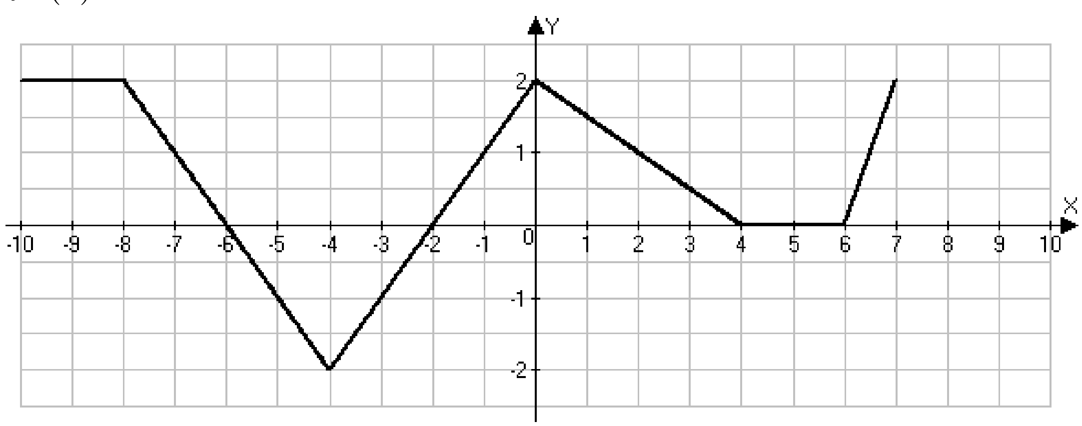
a. State a theorem that is needed to evaluating this limit. Why are the conditions of the theorem met?
b. Use the theorem to evaluate the limit.

9. Consider $\lim_{x\to 0}\frac{\sin(x)}{x}$ and assume you've established the inequality $\cos x \le \frac{\sin x}{x} \le 1$ for x near zero.

a. State a theorem that is useful to evaluating this limit. Why are the conditions of the theorem met?
b. Use the theorem to evaluate the limit.

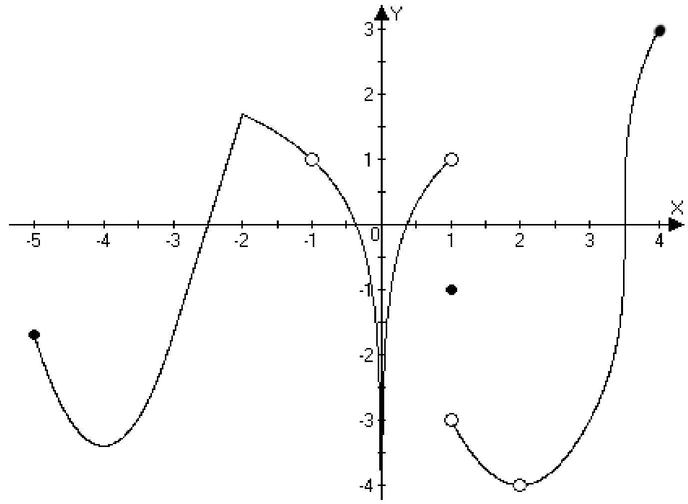
10. For the function $f(x)$ whose derivative function $f'(x)$ is graphed below, find where:

a. $f(x)$ is increasing
b. $f(x)$ has a local maximum.
c. $f''(x)$ is positive.
d. $f''(x)=0$.



Math 1A – Chapter 2 Test Solutions – Spring '08

1. The graph of $y = f(x)$ is given. Assume that $x = 0$ is a vertical asymptote



a. $\lim_{x\to -1} f(x) = 1$

b. $\lim_{x\to 1^+} f(x) = -3$

c. $\lim_{x\to 1} f(x)$ doesn't exist, since

$\lim_{x\to 1^+} f(x) = 1 \neq -3 = \lim_{x\to 1^-} f(x)$

d. $\lim_{x\to a} f(x)$ does not exist at $a = 0$ and $a = 1$.

e. For $\lim_{x\to a} f(x)$ to exist, with f discontinuous at a, you'd have what's called a removable discontinuity. There's one at (–1,1) and another at (2,–4) .

f. The derivative function $f'(x)$ has a jump discontinuity anywhere there's an instantaneous change in slope from one real number to another. This appears to happen where $x = -2$ and where $x = 1$.

2. Find the limit. Explain your answers.

a. $\lim_{x\to 3}\frac{(x-1)^2-1}{x^2-9} = \frac{3}{0^+} = \infty$

b. $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{(\sqrt{x}-1)}{(x-1)}\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)} = \lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{2}$

c. $\lim_{x\to\infty}\frac{\sin x}{1+\ln x} = 0$ by the squeeze theorem: $\frac{-1}{1+\ln x} \le \frac{\sin x}{1+\ln x} \le \frac{1}{1+\ln x}$ and $\lim_{x\to\infty}\frac{\pm 1}{1+\ln x} = \frac{\pm 1}{1+\infty} = 0$

d. $\lim_{x\to\infty} e^{\sin x}$ does not exist since it oscillates between e^{-1} and e.

3. Use the intermediate value theorem to prove that $\frac{18}{\pi^2}x^2 = \tan x$ has a solution in $\left(\frac{\pi}{6},\frac{\pi}{4}\right)$.

SOLN: Let $f(x) = \frac{18}{\pi^2}x^2 - \tan x$. Then f is continuous on $\left[\frac{\pi}{6},\frac{\pi}{4}\right]$. Now since 0 is between $f\left(\frac{\pi}{6}\right) = \frac{18}{\pi^2}\frac{\pi}{36}^2 - \tan\frac{\pi}{6} = \frac{1}{2} - \frac{1}{\sqrt{3}} < 0$ and $f\left(\frac{\pi}{4}\right) = \frac{18}{\pi^2}\frac{\pi}{16}^2 - \tan\frac{\pi}{4} = \frac{9}{8} - 1 > 0$, so by the Intermediate Value Theorem, there is c in $\left(\frac{\pi}{6},\frac{\pi}{4}\right)$ such that $f(c) = 0$ and this c proves the existence of a solution.

4. If the tangent to $y = f(x)$ at (0,3) passes through the point (2,0), find $f'(0)$.

SOLN: $f'(0) = \frac{0-3}{2-0} = -\frac{3}{2}$

5. Find the derivative function for $f(x)=\frac{1}{x+1}$ using the definition of the derivative.

SOLN: $$f'(x)=\lim_{h\to0}\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h}=\lim_{h\to0}\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h}\frac{(x+1)(x+h+1)}{(x+1)(x+h+1)}=\lim_{h\to0}\frac{(x+1)-(x+h+1)}{h(x+1)(x+h+1)}$$
$$=\lim_{h\to0}\frac{-h}{h(x+1)(x+h+1)}=\lim_{h\to0}\frac{-1}{(x+1)(x+h+1)}=\frac{-1}{(x+1)^2}$$

6. Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x)=\frac{1}{\sqrt{x^3+1}}$ for all x.
   1. Find an equation for the tangent line at (2,–4).
      SOLN: The slope is $g'(2)=\frac{1}{\sqrt{2^3+1}}=\frac{1}{3}$ so the tangent line is described by $y=-4+\frac{1}{3}(x-2)$
   2. Use the tangent line approximation to estimate $g(1.95)$ and $g(2.05)$.
      SOLN:
      $$g(1.95)\approx-4+\frac{1}{3}(1.95-2)=-4-\frac{.05}{3}=-4-\frac{1}{60}=-4.01\overline{6}$$
      $$g(2.05)\approx-4+\frac{1}{3}(2.05-2)=-4+\frac{.05}{3}=-4+\frac{1}{60}=-3.98\overline{3}$$
   3. Are your estimates in part (a) too large or too small? Explain.
      SOLN: Since the slopes $g'(x)=\frac{1}{\sqrt{x^3+1}}$ are decreasing in a neighborhood of 2, the function is concave down and so the tangent line is above the curve and these are overestimates.

7. Is there a number a such that $\lim_{x\to2}\frac{x-a}{x^2+x-6}$ exists? If not, why not? If so, find the value of a and the value of the limit.
   SOLN: $\lim_{x\to2}\frac{x-a}{x^2+x-6}=\lim_{x\to2}\frac{x-a}{(x+3)(x-2)}=\frac{1}{5}\Leftrightarrow a=2$

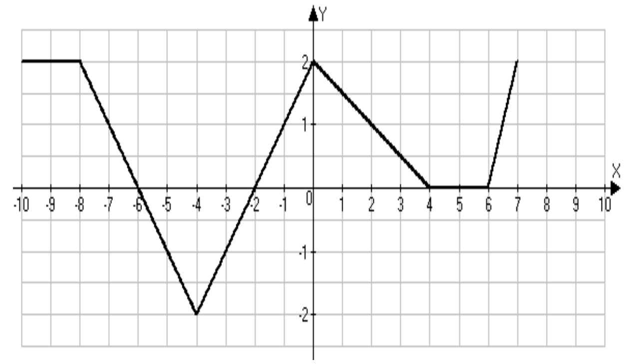
8. Consider $\lim_{x\to0}\arctan\left(x+e^x\right)$.
   a. Theorem: if $L=\lim_{x\to a}f(x)$ exists and g is continuous at L then $\lim_{x\to a}g\left[f(x)\right]=g\left[\lim_{x\to a}f(x)\right]=g(L)$
      In this case, all the functions involved are continuous everywhere, so the conditions of the theorem are met.
   b. $\lim_{x\to0}\arctan\left(x+e^x\right)=\arctan\left(\lim_{x\to0}x+\lim_{x\to0}e^x\right)=\arctan(1)=\frac{\pi}{4}$

9. Consider $\lim_{x\to0}\frac{\sin(x)}{x}$ and assume you've established the inequality $\cos x\le\frac{\sin x}{x}\le1$ for x near zero.

a. The squeeze theorem says that if $f(x) \le g(x) \le h(x)$ in some neighborhood of a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$. In this case, $1 = \lim_{x \to 0} 1 = \lim_{x \to 0} \cos(x)$

b. Thus $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

10. For the function $f(x)$ whose derivative function $f'(x)$ is graphed at right,



a. f is increasing on $(-10,-6) \cup (-2,4) \cup (6,7)$

b. f has a local maximum where x = -6

c. $f''(x)$ is positive on $(-4,0) \cup (6,7)$

d. $f''(x) = 0$ on $(-10,-8) \cup (4,6)$.