Stewart: Calculus, Concepts and Context, Exercise 4.4#6

Produce graphs of f that reveal all the important aspects of the curve. In particular, you should use graphs of f' and f'' to estimated the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points.

Define the function and its first two derivatives in Maple like so:

$f := x \rightarrow \tan(x) + 5 \cdot \cos(x);$

$$x \rightarrow \tan(x) + 5\cos(x)$$

$fprime(x) := diff(f(x), x);$

$$x \rightarrow \frac{d}{dx} f(x)$$

$fprime(a);$

$$1 + \tan(a)^2 - 5\sin(a)$$

$fpp(x) := diff(fprime(x), x);$

$$x \rightarrow \frac{d}{dx} fprime(x)$$

$fpp(a);$

$$2\tan(a)\left(1 + \tan(a)^2\right) - 5\cos(a)$$

Notice that setting the first derivative to zero and multiplying through by $\cos^2(x)$ yields

$1 - 5 \cdot \sin(x)\left(1 - \sin^2(x)\right);$

$$1 - 5\sin(x)\left(1 - \sin(x)^2\right)$$

$subs(\sin(x) = u, \%);$

$$1 - 5u\left(1 - u^2\right)$$

$solve\left(1 - 5u\left(1 - u^2\right) = 0, u\right);$

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Warning, solutions may have been lost
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What the heck kind of response is that? It's just a simple cubic, how could Maple have "lost" the solutions? Maybe if we expand and write in descending powers?

$solve\left(5 \cdot u^3 - 5 \cdot u + 1 = 0, u\right);$

$$\frac{1}{30}\left(-2700 + 300\,I\sqrt{219}\right)^{1/3} + \frac{10}{\left(-2700 + 300\,I\sqrt{219}\right)^{1/3}}, -\frac{1}{60}\left(-2700 + 300\,I\sqrt{219}\right)^{1/3} - \frac{5}{\left(-2700 + 300\,I\sqrt{219}\right)^{1/3}} + \frac{1}{2} I\sqrt{3}\left(\frac{1}{30}\left(-2700 + 300\,I\sqrt{219}\right)^{1/3} - \frac{10}{\left(-2700 + 300\,I\sqrt{219}\right)^{1/3}}\right), -\frac{1}{60}\left(-2700 + 300\,I\sqrt{219}\right)^{1/3} - \frac{5}{\left(-2700 + 300\,I\sqrt{219}\right)^{1/3}} - \frac{1}{2} I\sqrt{3}\left(\frac{1}{30}\left(-2700 + 300\,I\sqrt{219}\right)^{1/3} - \frac{10}{\left(-2700 + 300\,I\sqrt{219}\right)^{1/3}}\right)$$

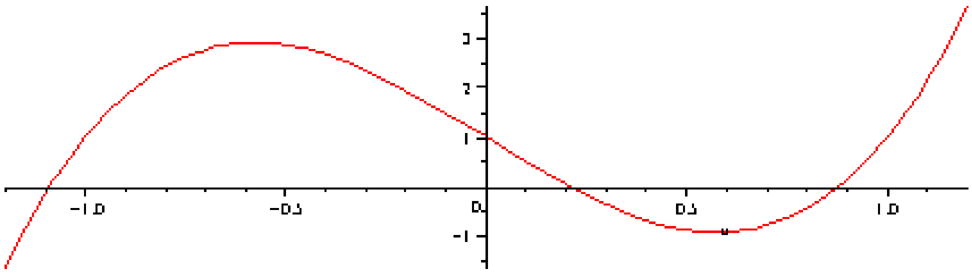
Right, so there you have them folks. Note that these are all written as imaginary numbers (Maple uses I for the imaginary unit.) Perhaps you'd care to approximate?

evalf(%);

$$0.8788850663 - 2.\ 10^{-10}\ \mathrm{I},\ -1.088033915 - 2.598076212\ 10^{-10}\ \mathrm{I},$$
$$0.2091488484 + 2.598076212\ 10^{-10}\ \mathrm{I}$$

The imaginary parts of these numbers look suspiciously like rounding errors. So what's going on? Are these, in fact, real numbers? Take a look at the graph:

$plot(5\cdot u^3 - 5\cdot u + 1, u = -1.2..1.2);$



Clearly, there are three real zeros and two of them are in the range of the sine function, so that's promising, but why are the zeros expressed in terms of the imaginary unit if they're real? Perhaps a little review of Tartaglia's method is in order.

We wish to solve

$u^3 - u = -\frac{1}{5};$

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Notice that, for any a and b,

$$(a-b)^3 + 3\,ab(a-b) = a^3 - b^3$$

so that if a and b satisfy $3\cdot ab = -1$ and $a^3 - b^3 = -\frac{1}{5}$ then $u = a - b$ is a solution to the equation.

Now, equating the coefficients of u we have $b = \frac{-1}{3\,a}$ and so $a^3 - b^3 = a^3 - \left(\frac{-1}{3\cdot a}\right)^3 = a^3 + \frac{1}{27\cdot a^3} = -\frac{1}{5}$

which (multiply through by a^3) is equivalent to $a^6 + \frac{1}{5}\cdot a^3 + \frac{1}{27} = 0$. This is quadratic in a^3 so we can complete the square and write the equivalent equation, $\left(a^3 + \frac{1}{10}\right)^2 = \frac{-1}{27} + \frac{1}{100}$.

Extracting roots, $a^3 + \frac{1}{10} = \pm\ \mathrm{sqrt}\left(\frac{-73}{2700}\right)$. Oh dear, a is not a real number. That doesn't mean the solutions are necessarily not reals, however. $a = \left(-\frac{1}{10} \pm i\ \mathrm{sqrt}\left(\frac{73}{2700}\right)\right)^{\frac{1}{3}}$ whence

$b = \frac{-1}{3}\left(-\frac{1}{10} \pm i\ \mathrm{sqrt}\left(\frac{73}{2700}\right)\right)^{-\frac{1}{3}}$ and so the solutions are

$\left(-\frac{1}{10} + \mathrm{I}\cdot \mathrm{sqrt}\left(\frac{73}{2700}\right)\right)^{\frac{1}{3}} + \frac{1}{3}\left(-\frac{1}{10} + \mathrm{I}\cdot sqrt\left(\frac{73}{2700}\right)\right)^{-\frac{1}{3}};$

$$\left(-\frac{1}{10} + \frac{1}{90}\,\mathrm{I}\sqrt{219}\right)^{1/3} + \frac{1}{3\left(-\frac{1}{10} + \frac{1}{90}\,\mathrm{I}\sqrt{219}\right)^{1/3}}$$

simplify(%);

$$\frac{1}{30}\,\frac{\left(-2700+300\,\mathrm{I}\,\sqrt{219}\right)^{2/3}+300}{\left(-2700+300\,\mathrm{I}\,\sqrt{219}\right)^{1/3}}$$

evalf(%);

$$0.8788850666+1.70803591010^{-10}\,\mathrm{I}$$

$$\left(-\frac{1}{10}-\mathrm{I}\cdot\mathrm{sqrt}\left(\frac{73}{2700}\right)\right)^{\frac{1}{3}}+\frac{1}{3}\left(-\frac{1}{10}-\mathrm{I}\cdot sqrt\left(\frac{73}{2700}\right)\right)^{-\frac{1}{3}};$$

$$\left(-\frac{1}{10}-\frac{1}{90}\,\mathrm{I}\,\sqrt{219}\right)^{1/3}+\frac{1}{3\left(-\frac{1}{10}-\frac{1}{90}\,\mathrm{I}\,\sqrt{219}\right)^{1/3}}$$

simplify(%);

$$\frac{1}{30}\,\frac{\left(-2700-300\,\mathrm{I}\,\sqrt{219}\right)^{2/3}+300}{\left(-2700-300\,\mathrm{I}\,\sqrt{219}\right)^{1/3}}$$

evalf(%);

$$0.8788850666-1.70803591010^{-10}\,\mathrm{I}$$

Hmm. Same number!

Now when the old Derive CAS is used to solve the equation with SOLVE(u^3-u-=1/5,u) we get

$$u=\frac{2\,\mathrm{sqrt}(3)\cdot\cos\left(\frac{\mathrm{arccot}\left(-\frac{3\,\mathrm{sqrt}(219)}{73}\right)}{3}\right)}{3};$$

$$u=\frac{2}{3}\sqrt{3}\,\sin\left(\frac{1}{6}\pi+\frac{1}{3}\,\mathrm{arccot}\left(\frac{3}{73}\sqrt{219}\right)\right)$$

evalf(%);

$$u=0.878885066'$$

$$u=-\frac{2\,\mathrm{sqrt}(3)\cdot\sin\left(\frac{\arctan\left(\frac{3\,\mathrm{sqrt}(219)}{73}\right)}{3}+\frac{\mathrm{Pi}}{3}\right)}{3}$$

$$u=-\frac{2}{3}\sqrt{3}\,\sin\left(\frac{1}{3}\arctan\left(\frac{3}{73}\sqrt{219}\right)+\frac{1}{3}\pi\right)$$

evalf(%);

$$u=-1.08803391:$$

$$u=\frac{2\,\mathrm{sqrt}(3)\cdot\sin\left(\frac{\arctan\left(\frac{3\,\mathrm{sqrt}(219)}{73}\right)}{3}\right)}{3};$$

$$u=\frac{2}{3}\sqrt{3}\,\sin\left(\frac{1}{3}\arctan\left(\frac{3}{73}\sqrt{219}\right)\right)$$

evalf(%);

$$u=0.209148848:$$

So, what's going on here?

What we need to do is work with the polar form of the complex values we get for *a* and *b*. So if

$a^3 = -\frac{1}{10} + \frac{\mathrm{sqrt}(219)}{90} \cdot I$

$$a^3 = -\frac{1}{10} + \frac{1}{90} I \sqrt{219}$$

Then the polar form of this complex number is the modulus times $e^{I\cdot\theta}$ where the modulus is

$\mathrm{sqrt}\left(\left(-\frac{1}{10}\right)^2 + \left(\frac{\mathrm{sqrt}(219)}{90}\right)^2\right);$

$$\frac{1}{9}\sqrt{3}$$

That's neat, huh? Now the angle θ is a little more messy. Since it's in the third quadrant,:

$\theta := \pi - \mathit{arctan}\left(\frac{\mathrm{sqrt}(219)}{90}\left(\frac{10}{1}\right)\right);$

$$\pi - \arctan\left(\frac{1}{9}\sqrt{219}\right)$$

Thus $a^3 = \frac{1}{9}\sqrt{3} \cdot \exp(I\cdot\theta);$

$$a^3 = \frac{1}{9}\sqrt{3}\, e^{I\left(\pi - \arctan\left(\frac{1}{9}\sqrt{219}\right)\right)}$$

Now by de Moivre's formula, $a := \left(\frac{1}{9}\sqrt{3}\right)^{\frac{1}{3}} \cdot \exp\left(\frac{I\cdot\theta}{3}\right);$

$$\frac{1}{9}\, 9^{2/3}\, 3^{1/6}\, e^{\frac{1}{3} I\left(\pi - \arctan\left(\frac{1}{9}\sqrt{219}\right)\right)}$$

simplify(%);

$$\frac{1}{3}\sqrt{3}\, e^{-\frac{1}{3} I\left(-\pi + \arctan\left(\frac{1}{9}\sqrt{3}\sqrt{73}\right)\right)}$$

Which makes $b := -\frac{1}{3\cdot a};$

$$-\frac{1}{9}\,\frac{9^{1/3}\, 3^{5/6}}{e^{\frac{1}{3} I\left(\pi - \arctan\left(\frac{1}{9}\sqrt{219}\right)\right)}}$$

simplify(%);

$$-\frac{1}{3}\sqrt{3}\, e^{\frac{1}{3} I\left(-\pi + \arctan\left(\frac{1}{9}\sqrt{3}\sqrt{73}\right)\right)}$$

whence $a - b;$

$$\frac{1}{9}\, 9^{2/3}\, 3^{1/6}\, e^{\frac{1}{3} I\left(\pi - \arctan\left(\frac{1}{9}\sqrt{219}\right)\right)} + \frac{1}{9}\,\frac{9^{1/3}\, 3^{5/6}}{e^{\frac{1}{3} I\left(\pi - \arctan\left(\frac{1}{9}\sqrt{219}\right)\right)}}$$

simplify(%);

$$\frac{2}{3}\sqrt{3}\,\sin\left(\frac{1}{6}\pi + \frac{1}{3}\arctan\left(\frac{1}{9}\sqrt{3}\sqrt{73}\right)\right)$$

evalf(%);

$$0.878885066'$$

You can find the other two zeros by adding $2 \cdot \pi \cdot k$ to θ before taking cube roots and setting $k = 1$ and then $k = 2$.

Here's my question, then: The definition of an algebraic number is a number that is a root of a polynomial equation with integer coefficients. If you can find the representation of an algebraic number it should be in terms of "surds" (radicals) and if they're real zeros, those expressions need not involve the imaginary unit. However, this example seems to suggest this is not so. Is that not so?