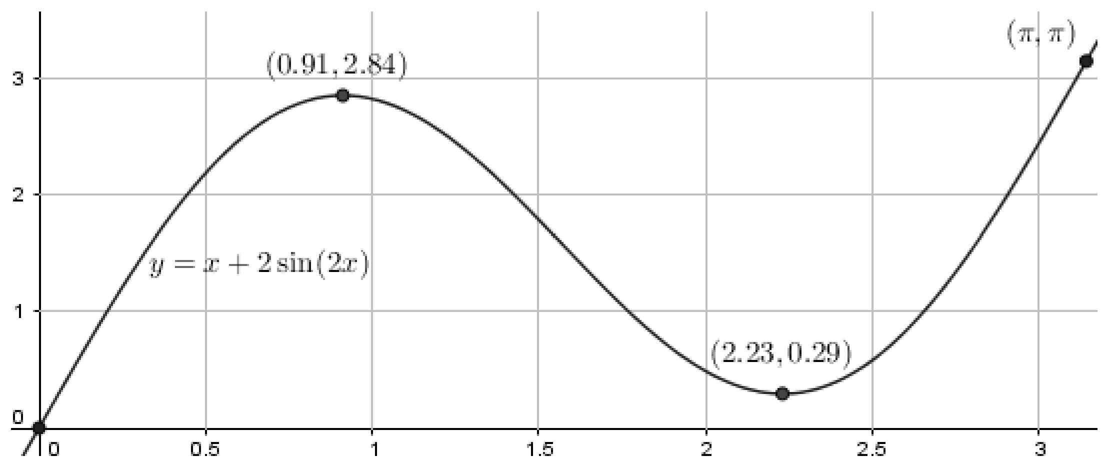
1. Find the local and absolute extreme values of the function $f(x) = x + 2\sin(2x)$ on the interval $[0, \pi]$.
ANS: The function is continuous on $[0, \pi]$ so we are the endpoints: $(0, f(0)) = (0, 0)$ and $(\pi, f(\pi)) = (0, \pi)$ are potential absolute maximum or minimum values. The function is also differentiable, so the only other critical points are where $f'(x) = 1 + 4\cos(2x) = 0$, that is, where $x = \frac{1}{2}\arccos(-\frac{1}{4})$ and its supplement, $x = \pi - \frac{1}{2}\arccos(-\frac{1}{4})$. We note that, using symmetry from the unit circle, $\arccos(-\frac{1}{4}) = \dfrac{\pi}{2} + \arcsin(\frac{1}{4}) \approx \dfrac{\pi}{2} + \dfrac{1}{4}$, where this last approximation comes from $\sin\theta \approx \theta \Leftrightarrow \theta \approx \arcsin\theta$, when θ is small. This allows a good approximation to the value of the function at the critical points: $f(\frac{1}{2}\arccos(-\frac{1}{4})) = \frac{1}{2}\arccos(-\frac{1}{4}) + 2\sin(\arccos(-\frac{1}{4})) \approx \dfrac{1}{2}\left(\dfrac{\pi}{2} + \dfrac{1}{4}\right) + \dfrac{\sqrt{15}}{2} \approx$ $(1.57 + 0.25)/2 + 2 = 2.91$, a little less. So this is a local maximum but less than the absolute maximum at (π, π). Similarly, $f(\pi - \frac{1}{2}\arccos(-\frac{1}{4})) = \pi - \frac{1}{2}\arccos(-\frac{1}{4}) + 2\sin(2\pi - \arccos(-\frac{1}{4})) \approx \pi - \dfrac{1}{2}\left(\dfrac{\pi}{2} + \dfrac{1}{4}\right) - \dfrac{\sqrt{15}}{2}$
$\approx 3.14 - 0.91 - 2 = 0.23$, or a little more. This is a local minimum, but not a global minimum. This is the Geogebra graph the confirms the above work:



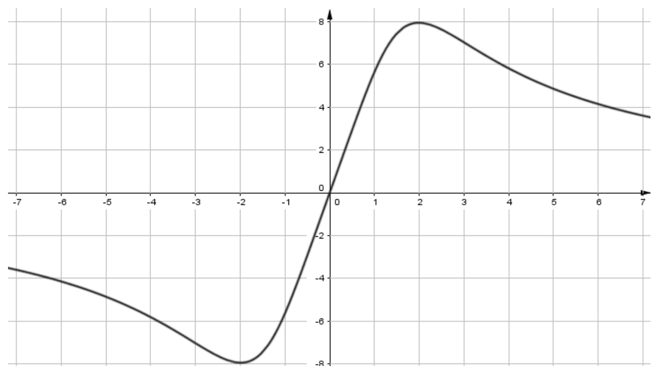
To be sure, we could improve our estimate of $\sqrt{15}$ using the linearization of $r(x) = \sqrt{x}$ near $x = 16$: $r(15) \approx r(16) + r'(16)(15 - 16) = 4 - \dfrac{1}{8} = 3.875$, so $\dfrac{\sqrt{15}}{2} \approx 1.9875$.
Also, we know that $\frac{1}{4} \approx \arcsin(\frac{1}{4})$ is a slight underestimate. It turns out that $\arcsin(\frac{1}{4}) \approx 0.2527$ is closer. So $\arccos(-\frac{1}{4}) \approx 1.572 + 0.253 = 1.825$, but our approximation was good enough to establish the global extrema.

2. Evaluate the limit.
   (a) $\displaystyle\lim_{x\to 0^+} \frac{\tan(x^2)}{x\sin(x)}$
   ANS: Both numerator and denominator approach 0, so this is a L'Hospital situation:
   $\displaystyle\lim_{x\to 0^+} \frac{\tan(x^2)}{x\sin(x)} = \lim_{x\to 0^+} \frac{2x\sec^2(x^2)}{\sin(x) + x\cos(x)}$
   Ooops...still both numerator and denominator are approaching zero...so we repeat:
   $\displaystyle= \lim_{x\to 0^+} \frac{2\sec^2(x^2) + 4x^2\sec^2(x^2)\tan(x^2)}{2\cos(x) - x\sin(x)} = 1$

   (b) $\displaystyle\lim_{x\to 0^+} (\cos(x))^{1/x}$
   ANS: Here the indeterminate form 1^∞, so we need to consider $\displaystyle\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \ln\cos(x))^{1/x} = \lim_{x\to 0^+} \frac{1}{x}\ln\cos(x)$ which leads to the indeterminate form $\dfrac{0}{0}$ which is a L'Hospital situation. Thus $\displaystyle\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{-\tan(x)}{1} =$ 0, which means that $\displaystyle\lim_{x\to 0^+} (\cos(x))^{1/x} = 1$

3. Sketch the graph of a function that satisfies the given conditions:
$f(0) = 0$,
f is odd and continuous,
$f'(x) > 0$ on $(0, 2)$,
$f'(x) < 0$ on $(2, \infty)$,
$f''(x) < 0$ on $(0, 3)$
$f''(x) > 0$ on $(3, \infty)$ and
$\lim\limits_{x\to\infty} f(x) = 0$.

The function shown at right meets these descriptors. It was created with Geogebra, entering the function $f(x) = \dfrac{\arctan(ax^2)}{bx}$ and then adjusting the parameters (using sliders) a and b to position the stationary and inflection points. Note that $\lim\limits_{|x|\to\infty} f(x) = 0$.



4. Suppose $f(x) = xe^{-x^2}$

   (a) What is the domain of f?
   ANS: Domain $= \mathbb{R}$

   (b) What intercept(s) does f have?
   ANS: The only intercept is $(0, 0)$

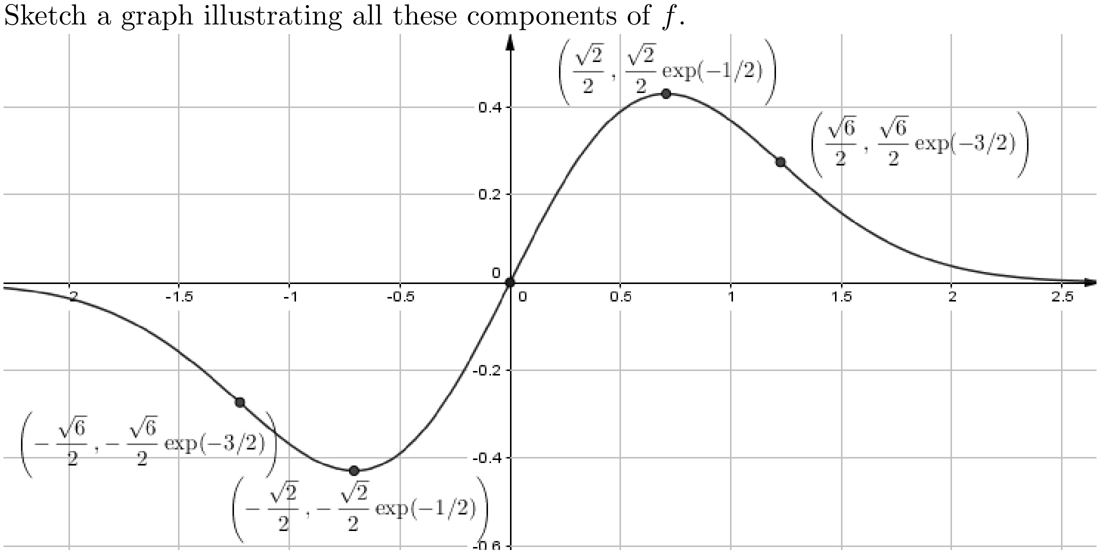
   (c) Describe the symmetry of f.
   $f(-x) = -xe^{-(-x)^2} = -f(x)$, so this function has odd (origin) symmetry.

   (d) What asymptote(s) does f have?
   $\lim\limits_{|x|\to\infty} f(x) = 0$, so the x-axis is a horizontal asymptote.

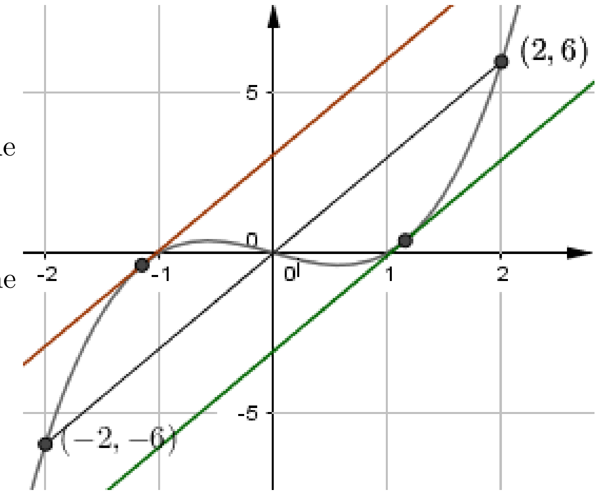
   (e) Simplify $f'(x)$ and use it to find critical numbers and intervals of increase/decrease.
   $f'(x) = e^{-x^2}(1-2x^2) > 0 \Leftrightarrow x \in \left(\dfrac{-\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ so f is decreasing on $\left(-\infty, \dfrac{\sqrt{2}}{2}\right) \cup \left(\dfrac{\sqrt{2}}{2}, \infty\right)$ and decreasing in between those intervals.

   (f) Simplify $f''(x)$ and use that to find inflection points of f.
   $f''(x) = e^{-x^2}(4x^3 - 6x) > 0 \Leftrightarrow x \in \left(\dfrac{-\sqrt{6}}{2}, 0\right) \cup \left(\dfrac{\sqrt{6}}{2}\right)$

   (g) Sketch a graph illustrating all these components of f.

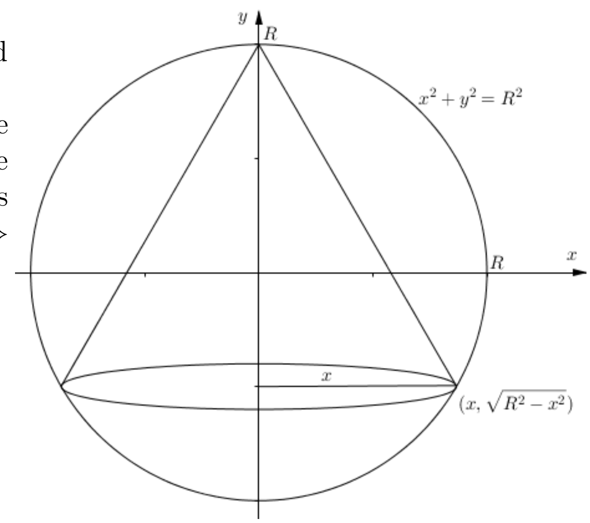


5. Consider $g(x) = x^3 - x$ on $[-2, 2]$
   (a) Explain why this function satisfies the premis of the Mean Value Theorem on that interval.
   ANS: g is differentiable on $[-2, 2]$
   (b) Find all values of $c \in (-2, 2)$ that safisfy the conclusion of the Mean Value Theorem.
   We solve $g'(c) = \dfrac{g(2) - g(-2)}{2 - (-2)} \Leftrightarrow 3c^2 - 1 = \dfrac{12}{4} \Leftrightarrow c = \pm\dfrac{2\sqrt{3}}{3}$



6. Find the minimum value of $\ln(x^2 + y)$ so that $x + y = 5$.
ANS: We impose the constraint by substituting into the object function to obtain a function of a single variable: $f(x) = \ln(x^2 - x + 5)$ and then set its derivative to zero to find the minimum value: $f'(x) = \dfrac{2x - 1}{x^2 - x + 5} = 0$ when $x = \frac{1}{2}$. Now f' changes from negative to positive as x increases through $\left(\frac{1}{2}, \frac{19}{4}\right)$, so $f(\frac{1}{2}) = \frac{19}{4}$ is a minimum.

7. (12 points) Find the volume of the largest cone that can be inscribed in a sphere of radius R.
ANS: The cross-section of the sphere with the xy-plane gives the equation $x^2 + y^2 = R^2$, so that the square of the radius of the base of the cone is $x^2 = R^2 - y^2$ and the volume of the cone is $V = \frac{1}{3}\pi x^2(R+y) = \frac{1}{3}\pi(R^2 - y^2)(R+y) = \frac{1}{3}\pi(R^3 + R^2y - Ry^2 - y^3) \Rightarrow$
$V'(y) = \frac{1}{3}\pi(R^2 - 2Ry - 3y^2) = 0 \Leftrightarrow 3\left(y + \dfrac{R}{3}\right)^2 = \dfrac{4R^2}{3}$
So $|y| = \dfrac{R}{3}$ and the maximum volume is $V = \frac{1}{3}\pi\left(\dfrac{8R^2}{9}\right)\left(\dfrac{4R}{3}\right)$
$\boxed{= \dfrac{32\pi R^3}{81}}$



8. (16 points) To find the cube root of a number, A, we want to find a zero of the polynomial $p(x) = x^3 - A$
   (a) Show that Newton method for finding the cube root of A is equivalent to iterating

$$x_{n+1} = \frac{2x_n + \dfrac{A}{x_n^2}}{3}$$

   ANS: Newton's method has $x_{n+1} = x_n - \dfrac{p(x_n)}{p'(x_n)}$. In this case, $p'(x_n) = 3x_n^2$, so $x_{n+1} = x_n - \dfrac{x_n^3 - A}{3x_n^2}$ and getting a common denominator yields $x_{n+1} = \dfrac{3x_n^3 - x_n^3 + A}{3x_n^2} = \dfrac{2x_n + A/x_n^2}{3}$, as desired.
   (b) Starting with $x_1 = 1$, find x_2 and x_3 in the Newton's method search for $\sqrt[3]{3}$
   $x_2 = \dfrac{2 + 3/1}{3} = \dfrac{5}{3} = 1.\overline{6}$
   $x_3 = \dfrac{10/3 + 3/(25/9)}{3} = \dfrac{10/3 + 27/25}{3} = \dfrac{250 + 81}{225} = \dfrac{331}{225}$ Note that $\sqrt[3]{3} \approx 1.44224$ and $\dfrac{331}{225} \approx 1.47\overline{1}$