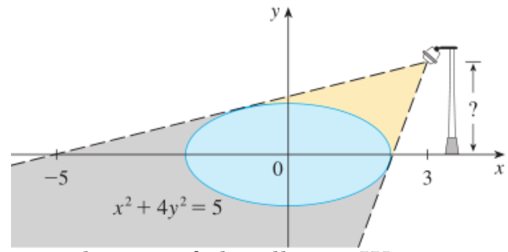
1. The figure shows a lamp located three units to the right of the y-axis and a shadow created by the elliptical region $x^2 + 4y^2 \leq 5$. If the point $(-5, 0)$ is on the edge of the shadow, how far above the x-axis is the lamp located?



ANS: There are two ways to express the slope of the line tangent to the top of the ellipse. We set up an equation by saying these are equal. One expression may be obtained by differentiating implicitly and solving for y': $2x+8yy' = 0 \Leftrightarrow y' = -\dfrac{x}{4y} = -\dfrac{x}{2\sqrt{5-x^2}}$. The other expression is just to plug the points $(-5, 0)$ and $(x, \frac{1}{2}\sqrt{5-x^2}$ into the slope formula: $y' = \dfrac{y}{x+5} = \dfrac{\frac{1}{2}\sqrt{5-x^2}}{x+5}$. The expressions involving both x and y can be equated like so: $-\dfrac{x}{4y} = \dfrac{y}{x+5} \Leftrightarrow -x^2 - 5x = 4y^2$, and substituting $y^2 = \frac{1}{4}(5 - x^2)$ yields $-x^2 - 5x = 5 - x^2 \Leftrightarrow x = -1 \Leftarrow y = 1$. Thus the slope of the tangent line is $y' = \frac{1}{4}$, whence the equation $y - 1 = \frac{1}{4}(x + 1) \Leftrightarrow y = \frac{1}{4}x + \frac{5}{4}$, so, at the lamppost, $y = 2$

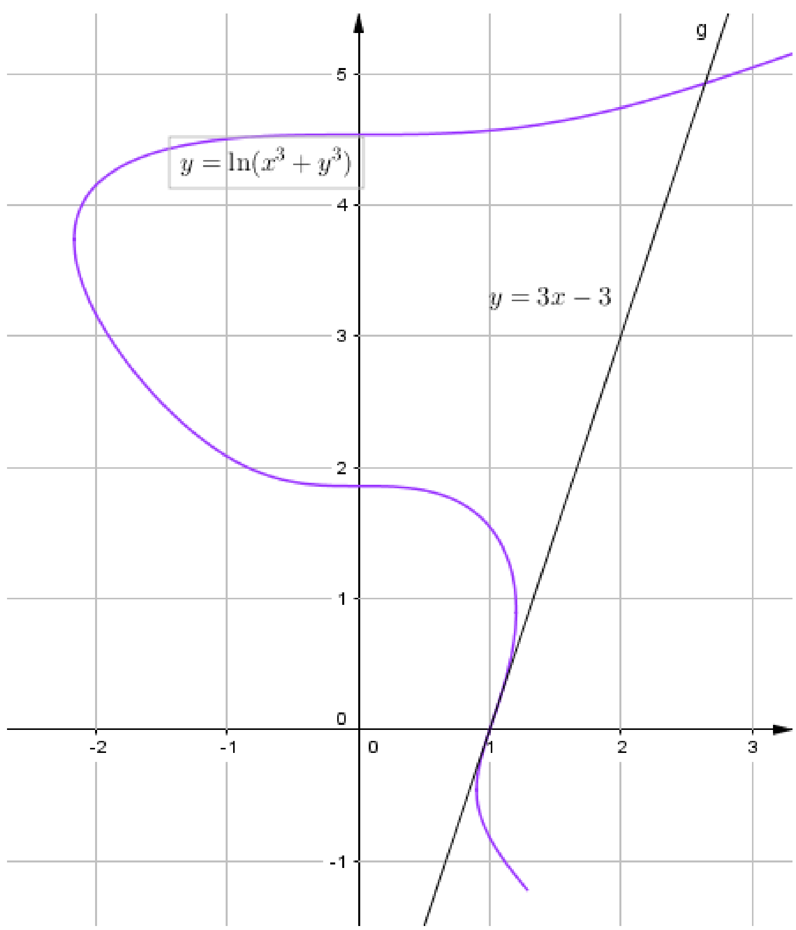
2. Consider the equation $y = \ln(x^3 + y^3)$

   (a) Find a formula for $\dfrac{dy}{dx}$ in terms of x and y.

   ANS: $\dfrac{dy}{dx} = \dfrac{3x^2 + 3y^2\frac{dy}{dx}}{x^3 + y^3} \Leftrightarrow (x^3 + y^3)\frac{dy}{dx} = 3x^2 + 3y^2\frac{dy}{dx} \Leftrightarrow (x^3 + y^3 - 3y^2)\frac{dy}{dx} = 3x^2 \Leftrightarrow \frac{dy}{dx} = \dfrac{3x^2}{y^3 + x^3 - 3y^2}$

   (b) Find an equation for the line tangent to the curve at $(1, 0)$.

   ANS: $\frac{dy}{dx} = \dfrac{3}{1} = 3$, so the equation is $y = 3(x - 1)$. As a bonus, we can visualize the implicit curve and its tangent line using Geogebra:



3. A ladder 5 meters long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 0.1 m/s,

   (a) How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 meters from the wall?
   ANS: The ladder, wall and ground form a right triangle, so $x^2 + y^2 = 25$. Differentiating implicitly with respect to time, t, gives $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. We're given that $\frac{dx}{dt} = 0.1$, so $\frac{dy}{dt} = -0.1\frac{x}{y}$. When $x = 3, y = \sqrt{25-9} = 4$, so the top of the ladder is sliding down at a rate of $\dfrac{3}{40}$ m/s, a little faster than the bottom is sliding out.

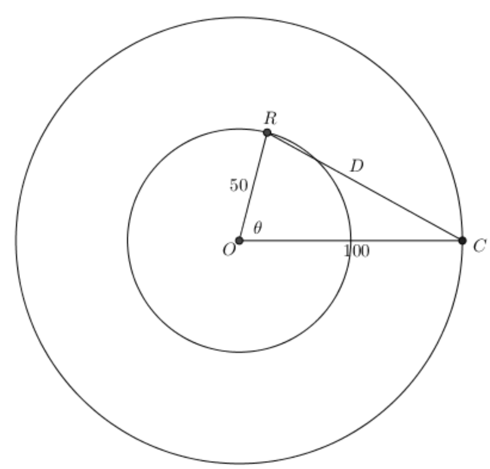
   (b) How fast is the angle between the ladder and the ground changing when the bottom of the ladder is 3 meters from the wall?
   ANS: $tan\theta = \dfrac{y}{x}$ and differentiating implicitly with respect to time gives,

$$\sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2} = \frac{3\frac{3}{40} + 4\frac{1}{10}}{9} = \frac{25}{9\cdot 40}$$

   $\sec(\theta) = \sec(\arctan(4/3)) = \dfrac{5}{3}$. Putting these together, the rate of change of the angle is
   $\dfrac{d\theta}{dt} = \dfrac{25}{9\cdot 40}\cdot\dfrac{9}{25}\boxed{= \dfrac{1}{40}\text{ rad/sec.}}$
   However, it would be simpler to start with the relation $\cos\theta = \dfrac{x}{5}$ and proceed from there.

4. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant the volume is $8\texttt{m}^3$, the pressure is 100 kPa, and the pressure is increasing at a rate of `5kPa/min`. At what rate is the volume decreasing at this instant?
   ANS: Differentiating implicity wrt time, $P\dfrac{dV}{dt} + V\dfrac{dP}{dt} = 0$. Substituting the given values, yields $100\dfrac{dV}{dt} + 8(5) = 0 \Leftrightarrow \dfrac{dV}{dt} = -\dfrac{2}{5}$ cubic meters per min.

5. A runner sprints around a circular track of radius 50 m at a constant speed of 8 m/s. The runner's coach is standing at a distance 100 m from the center of the track. Note: this means anywhere on a circle of radius 100 m concentric with the track. How fast is the distance between them changing when the distance between them is 100 m? Start by drawing a diagram.
   ANS: Refer to the diagram at right, where, without loss of generality, the Coach is placed a C and the runner is running around the track counterclockwise at R. By definition, $\theta = \dfrac{\texttt{arclength}}{50} \Leftrightarrow \dfrac{d\theta}{dt} = \dfrac{8}{50} = \dfrac{4}{25}$ rad/sec. From the law of cosines, $D^2 = 100^2 + 50^2 + 2(100)(50)\cos\theta \Rightarrow 2D\dfrac{dD}{dt} = -10000\sin\theta\dfrac{d\theta}{dt} \Leftrightarrow \dfrac{dD}{dt} = -8\sin\theta$. When $D = 100$, $\triangle ROC$ is isosceles, so $\cos\theta = \dfrac{25}{100} = \dfrac{1}{4} \Rightarrow \sin\theta = \dfrac{\sqrt{15}}{4}$ so $\dfrac{dD}{dt} = \dfrac{8\sqrt{15}}{4}$ m/sec.



6. Consider the function $f(x) = \dfrac{1}{x}$

   (a) Find a linearization near the point where $x = 4$.
   ANS: $f(x) \approx f(4) + f'(4)(x-4) = \frac{1}{4} - \frac{1}{16}(x-4)$

   (b) Use your linearization to approximate $\dfrac{1}{3.98}$
   ANS: $\dfrac{1}{3.98} \approx \frac{1}{4} - \frac{1}{16}(-0.02) = \frac{1}{4} + \frac{1}{800} = \frac{201}{800}$