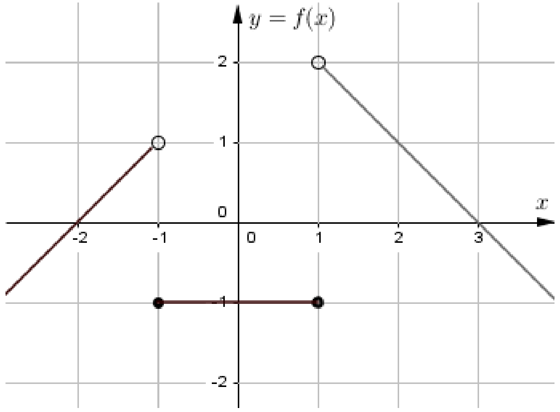
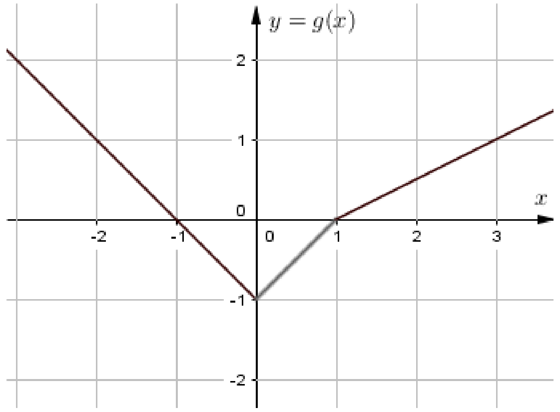
1. Consider the functions $y = f(x)$ and $y = g(x)$ defined by their graphs below.





(a) Find each limit or write "DNE" if the limit does not exist.

i $\lim_{x\to -1^+} f(x) = -1$

ii $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$

iii $\lim_{x\to -1} f(x)g(x) = 0$

iv $\lim_{x\to -2} g(f(x)) = -1$

v $\lim_{x\to -1} f(x) + g(x) = DNE$

vi $\lim_{x\to 1^+} 2f(x) + 3g(x) = 4$

(b) Find all the discontinuities of $y = f(x)$ and classify each as either a removable discontinuity, a jump discontinuity or a vertical asymptote.
ANS: $f(x)$ has jump discontinuities at $x = \pm 1$.

(c) Find all discontinutities of $y = g'(x)$.
$g(x)$ abruptly changes slope (i.e. $g'(x)$ has a discontinuity) at where $x = 0, 1$.

(d) Find $\frac{d}{dx} g(f(x))$ at $x = 2$.

$$\frac{d}{dx} g(f(x))\Big|_{x=2} = f'(2)\cdot g'(f(2)) = (-1)g'(1) \text{ is undefined, since } \lim_{x\to 1^-} g'(x) = 1 \neq 2 = \lim_{x\to 1^+} g'(x)$$

2. Use the **definition** of the derivative (that is, $f'(x) \equiv \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$) to show that $\frac{d}{dx}\sin(2x) = 2\cos(2x)$.

SOLN: $\lim_{h\to 0} \frac{\sin(2x+2h)-\sin(2x)}{h} = \lim_{h\to 0} \frac{\sin(2x)\cos(2h)+\sin(2h)\cos(2x)-\sin(2x)}{h}$

$= \lim_{h\to 0} \frac{\sin(2x)(\cos(2h)-1)}{h} + \frac{\cos(2x)\sin(2h)}{h} = 2\lim_{2h\to 0} \frac{\sin(2x)(\cos(2h)-1)}{2h} + \frac{\cos(2x)\sin(2h)}{2h}$

$= 2\sin(2x)\lim_{2h\to 0} \cancelto{0}{\frac{\cos(2h)-1}{2h}} + 2\cos(2x)\lim_{2h\to 0} \cancelto{1}{\frac{\sin 2h}{2h}} = 2\cos(2x)$

3. Use the Intermediate Value Theorem to prove that the equation $x^2 = \cos(x)$ has a solution in the interval $[0, \pi]$. Be sure to state how some function satisfies the conditions of the theorem before your conclusion.
SOLN: Let $f(x) = x^2 - \cos(x)$. Then $f(x)$ is continuous (a difference of continuous functions is continuous) so it satisfies the premis of the Intermediate Value Theorem on $[0, \pi]$. Also $f(0) = -1$ and $f(\pi) = \pi^2 + 1 > 0$ so, by the IVT, there exists some $c \in (0, \pi)$ such that $f(c) = c^2 - \cos(c) = 0 \Leftrightarrow c^2 = \cos(c)$.

4. A balloon is rising vertically and is observed from a point on the flat ground 100 meters from the spot directly beneath the balloon. At what rate is the balloon rising when the angle between the ground and the observer's line of sight is $\frac{\pi}{4}$ and is increasing at a rate of $\frac{\pi}{60}$ radians per second?

ANS: Let $h(t)$ = the height of the balloon at time t. Then $\tan\theta = \frac{h(t)}{100} \Rightarrow \frac{d}{dt}\tan\theta = \frac{h'(t)}{100}$

$\Leftrightarrow h'(t) = 100\sec^2\theta \frac{d\theta}{dt} = 100\sec^2\left(\frac{\pi}{4}\right)\cdot\frac{\pi}{60} = \frac{10\pi}{3} \approx 10\text{m/s}.$

5. Use a linear approximation to estimate $\sqrt[10]{0.97}$.
ANS: Let $f(x) = \sqrt{x}$. Then the line tangent to $f(x)$ at $x = 1$ is $L(x) = f(1) + f'(1)(x-1) = 1 + \frac{1}{10}(x-1)$
Thus $f(0.97) \approx L(0.97) = 1 + \frac{0.97-1}{10} = 1 - 0.003 = 0.997$, calculator gives $\sqrt[10]{0.97} \approx 0.9969587$

6. Use L'Hopital's rule, if appropriate, in the following:

   (a) Evaluate $\lim\limits_{x\to 0}\dfrac{\ln(1+2x)}{3x} = \boxed{\lim\limits_{x\to 0}\dfrac{2}{3(1+2x)} = \frac{2}{3}}$

   (b) Evaluate $\lim\limits_{x\to 0^+}\dfrac{\tan(2x)}{x} = \boxed{\lim\limits_{x\to 0}\dfrac{2\sec^2(2x)}{1} = 2}$

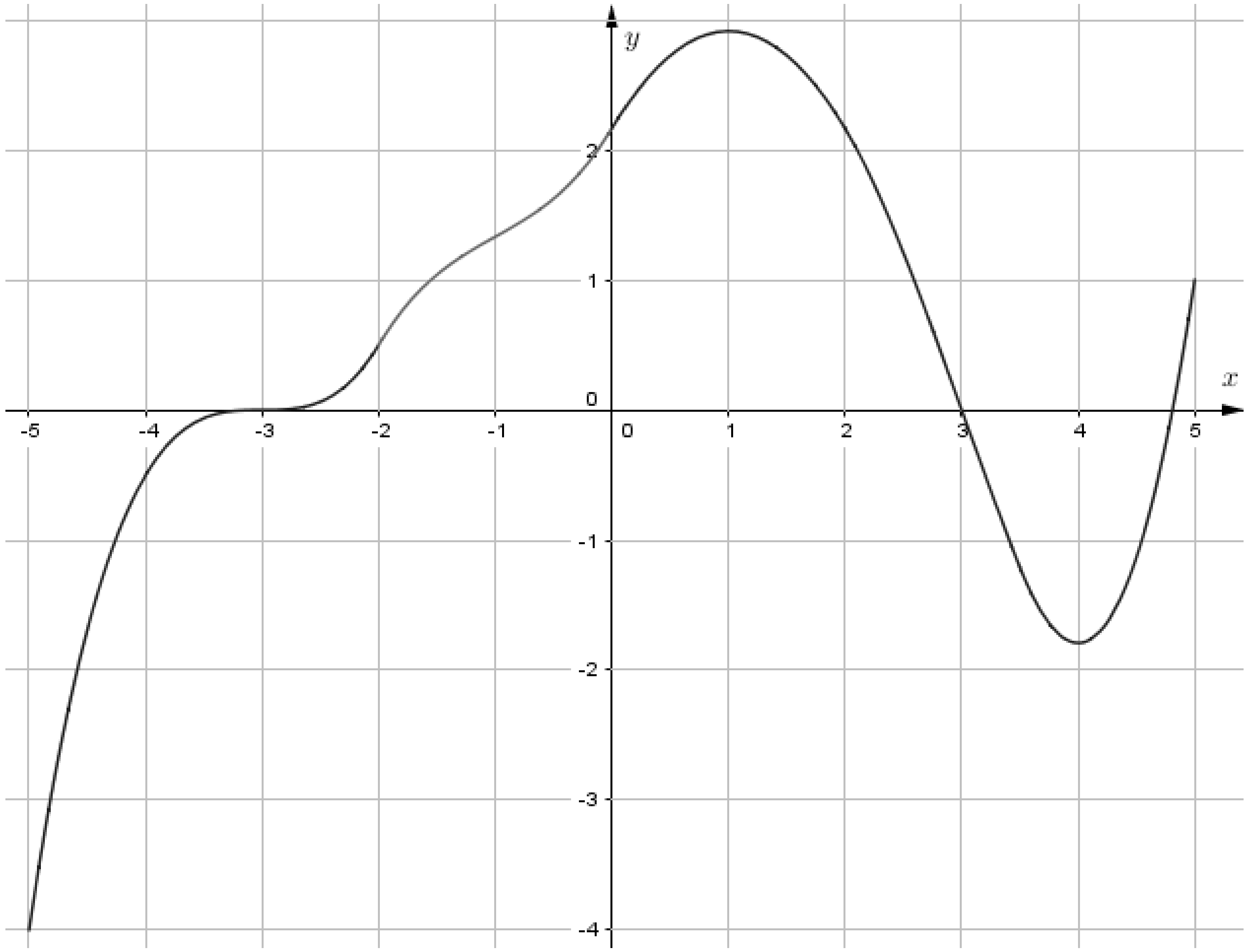
   (c) Suppose that f and f' are continuous functions and $f(0)=0$. If we know that

$$\lim_{x\to 0}\frac{f(x)}{\sin(2x)} = 3,$$

   evaluate $f'(0)$.
   ANS: Since the denominator is going to 0 and the limit exists, the numerator must also be going to 0 (so that didn't have to be given.) L'Hospital's rule gives $\lim\limits_{x\to 0}\dfrac{f(x)}{\sin(2x)} = \lim\limits_{x\to 0}\dfrac{f'(x)}{2\cos(2x)} = \frac{1}{2}\cdot f'(0) = 3$
   Thus $f'(0) = 6$

7. Assume that f is a function defined on the interval $[-5,5]$ with $f(-5)=-4$ and $f(5)=1$. Also, assume that f' and f'' exist and are continuous on $(-5,5)$. Use the information in the tables below to sketch a possible graph of f.

x	$-5\le x<-3$	-3	$-3<x<-1$	-1	$-1<x<1$	1	$1<x<3$	3	$3<x<4$	4	$4<x\le 5$
$f'(x)$	$+$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
$f''(x)$	$-$	0	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$



The piecewise function used to create this graph is (fwiw):

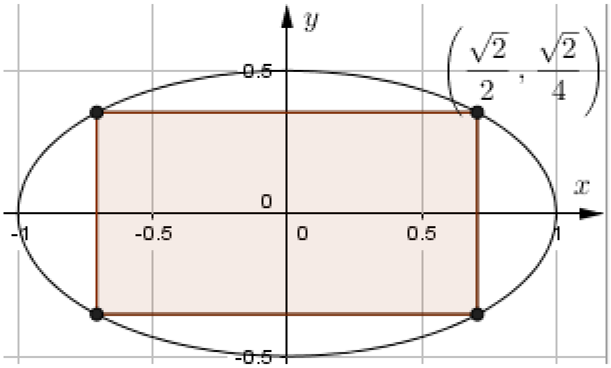
$$f(x) = \begin{cases} \frac{1}{2}(x+3)^3 & : -5 \le x < -2 \\ \frac{1}{3}(x+1)^3 + \frac{1}{2}(x+1) + \frac{4}{3} & : -2 \le x < 0 \\ \frac{35}{12} - \frac{3}{4}(x-1)^2 & : 0 \le x < 2 \\ \frac{1}{3}(x-3)^3 - \frac{5}{2}(x-3) & : 2 \le x < 3.5 \\ \frac{25}{81}(x-4)^3 + \frac{67}{27}(x-4)^2 - \frac{145}{81} & : 3.5 \le x \le 5 \end{cases}$$

8. Find all local and global maxima and minima of each of the following functions.
   (a) $f(x) = 2x^3 - 9x^2 + 12x + 1$ on the interval $[0, 3]$.
   ANS: This is a continuous, differentiable function so we can apply the closed interval method. At the endpoints we have $f(0) = 1$ and $f(3) = 10$ where as $f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$ has stationary points at $f(1) = 6$ and $f(2) = 5$, so the global min is (0,1) and the global max is (3,10).
   (b) $f(x) = \dfrac{1}{(x-2)^2}$ on the interval $[0, 3]$.
   ANS: This function is not continuous on $[0, 3]$ and $\lim\limits_{x \to 2} = \infty$, however it has a global min at $f(0) = \frac{1}{4}$.

9. Use the methods of constrained optimization to find the dimensions of the rectangle of largest area that can be inscribed in the ellipse, $x^2 + 4y^2 = 1$.

   ANS: Let $(x, y) = (\cos(t), \frac{1}{2}\sin(t))$ be the upper right corner of the rectangle. Then the area of the rectangle is $A(t) = 2\cos(t)\sin(t) = \sin(2t)$. Since $A'(t) = 2\cos(2t) = 0$ where $t = \pi/4$ the rectangle with the maximum area will have the upper right corner at $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{4}\right)$.

   It's dimensions are width $= \sqrt{2}$ and height $= \dfrac{\sqrt{2}}{2}$.



10. Consider the integral function $F(x) = \displaystyle\int_0^x 1 - t \, dt$
    (a) Evaluate $F(1)$ using the Fundamental Theorem of Calculus.
    $F(1) = \displaystyle\int_0^1 1 - t \, dt = t - \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}$
    (b) Find $F'(x)$ using the Fundamental Theorem of Calculus.
    ANS: $F'(x) = 1 - x$
    (c) Approximate $F(1)$ by a Riemann sum of $n = 4$ rectangles of equal width and right endpoints as sample points.
    ANS: $\Delta x = \dfrac{1-0}{4} = \frac{1}{4}$, $x_i = \dfrac{i}{4} \Rightarrow R_4 = \frac{1}{4}\sum\limits_{i=1}^{4} f(x_i) = \frac{1}{4}\sum\limits_{i=1}^{4}\left(1 - \dfrac{i}{4}\right) = \frac{1}{4}\left(\frac{3}{4} + \frac{1}{2} + \frac{1}{4} + 0\right) = \frac{3}{8}$