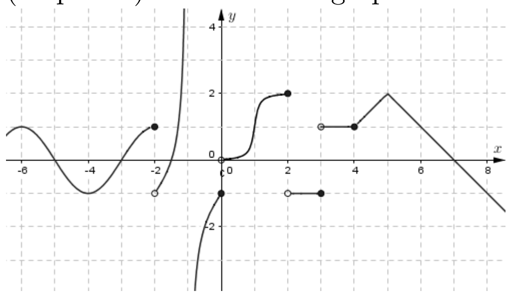
Math 1A - Spring '17Chapter 2 Test Solutions

Write all responses on separate paper. Show your work in detail for credit. No calculators.

1. (18 points) Consider the graph of the function $y = f(x)$ shown below:



i. $f(-2)\boxed{=1}$

ii. $\lim\limits_{x\to -2^+} f(x)\boxed{=-1}$

iii. $\lim\limits_{x\to 0^-} f(x)\boxed{=-1}$

iv. $\lim\limits_{x\to 2} f(x)\boxed{\text{DNE (jump)}}$

v. $\lim\limits_{x\to 2^+} f'(x)\boxed{=0}$

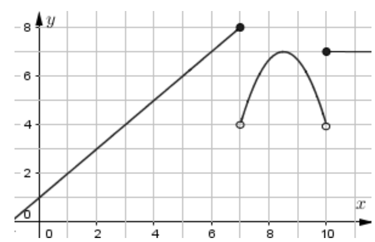
vi. $\lim\limits_{x\to 3} f'(x)\boxed{=0}$, $f'(3)$ DNE

(a) Find the limit, or explain why it does not exist.

(b) Is $f'(x)$ discontinuous where $x = 5$? Justify your answer using the definition of continuity.
SOLN: It appears as though $f(5) = 2$, $\lim\limits_{x\to 5^-} f(x) = 2$ and $\lim\limits_{x\to 5^+} f(x) = 2$ so f satisfies the conditions for continuity where $x = 5$, but $\boxed{f'(x) \text{ appears discontinuous since } \lim\limits_{x\to 5^-} f'(x) = 1 \neq \lim\limits_{x\to 5^+} f'(x) = -1}$

(c) Assume that f has a vertical asymptote along $x = -1$ as suggested by the graph. Is $\lim\limits_{x\to -1} f'(x) = \infty$? Why or why not?
ANS: Yes, since $\lim\limits_{x\to -1^-} f'(x) = \infty$ and $\lim\limits_{x\to -1^+} f'(x) = \infty$

$y = g(x)$:



2. (12 points) Suppose

$$f(x) = \begin{cases} x+8 & \text{if } x < 2 \\ x^2+3 & \text{if } x > 2 \end{cases}$$

and that $g(x)$ is defined by the graph shown at right.

(a) What is $\lim\limits_{x\to 2^-} f(x)$ $\boxed{\text{ANS: 10}}$

(b) What is $\lim\limits_{x\to 2^+} f(x)$ $\boxed{\text{ANS: 7}}$

(c) What is $\lim\limits_{x\to 2} g(f(x))$
ANS: As $x \to 2^-$, $f(x) \to 10^-$ so $g(f(x)) \to 4^+$. Similarly, as $x \to 2^+$, $f(x) \to 7^+$ so $g(f(x)) \to 4^+$. So from either side, the limit exists and is equal to the value of the function at the point.

(d) Can you apply the theorem that concludes that
$\lim\limits_{x\to 2} g(f(x)) = g\left(\lim\limits_{x\to 2} f(x)\right)$ Why or why not?
ANS: The theorem reguires that $f(x)$ be continuous where $x = 2$, so it does not apply.

3. (8 points) Let $P(t)$ = the inches of precipitation at Fantasy Springs on day t where t = the number of days since 1/1/2017. The table at right shows the value of this function over a 5 day period.

Date	$P(t)$ (in.)
1/1/2017	0.9
1/2/2017	1.1
1/3/2017	1.2
1/4/2017	0.9
1/5/2017	1.6

(a) Use the table to find the average rate of change in precipitation between 1/1/2017 and 1/5/2017. Be sure to specify the units of measure for this rate of change.
ANS: Average rate of change of P on $[0, 4]$ is $\dfrac{P(4) - P(0)}{4 - 0} = \dfrac{1.6 - 0.9}{4} = \dfrac{7}{40} \approx 0.17$ inches per day.

(b) What is your best approximation, based on this table, for rate of change on January 3?
ANS: The previous answer is one approximation for the rate of change on January 3, but a better approximation is $\dfrac{0.9 - 1.1}{3 - 1} = -0.1$ inches per day.

4. (18 points) Let $f(x) = \sqrt{x}$

   (a) Find the largest δ so that $|x-4| < \delta$ guarantees that $|f(x) - 2| < \frac{1}{2}$

   ANS: $|\sqrt{x}-2| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < \sqrt{x}-2 < \frac{1}{2} \Leftrightarrow \frac{3}{2} < \sqrt{x} < \frac{5}{2} \Leftrightarrow \frac{9}{4} < x < \frac{25}{4} \Leftrightarrow -\frac{7}{4} < x-4 < \frac{9}{4} \boxed{\Leftarrow |x-4| < \frac{7}{4}}$

   So the largest value for δ is $\boxed{\delta = \frac{7}{4}}$

   (b) Use the definition of the derivative as a limit to find the derivative function $f'(x)$.

   ANS: $f'(x) = \lim_{a\to x} \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim_{a\to x} \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} = \lim_{a\to x} \frac{1}{(\sqrt{x}+\sqrt{a})} = \boxed{\frac{1}{2\sqrt{x}}}$

   (c) Find an equation for the line tangent to $y = f(x)$ at $x = 4$.

   ANS: Plug into the point-slope formula: $y - f(4) = f'(4)(x-4) \Leftrightarrow y - 2 = \frac{1}{4}(x-4) \Leftrightarrow \boxed{y = \frac{1}{4}x + 1}$

5. (6 points) Find the smallest value of N so that if $x > N$ then $\frac{\pi}{2} - \arctan(x) < \frac{\pi}{6}$

   SOLN: We want $\arctan(x) > \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \Leftrightarrow x > \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, since $\arctan(x)$ is an increasing function of x. Thus $N = \sqrt{3}$ is the smallest value.

6. (8 points) Let

$$f(x) = \begin{cases} ax + b & : x < 1 \\ 3 & : x = 1 \\ bx - a & : x > 1 \end{cases}$$

   Find values of a and b so that f is a continuous function.

   SOLN: First observe that $\lim_{x\to 1^-} f(x) = a + b$ and so continuity requires $a + b = 3$. Now $\lim_{x\to 1^+} f(x) = b - a$ so we require $b - a = 3$. Solving the 2×2 linear system produces $\boxed{a = 0 \text{ and } b = 3}$.

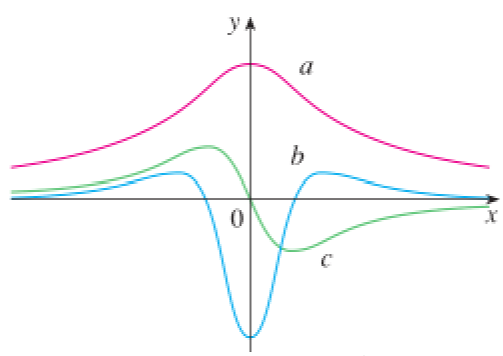
7. (10 points) Use the intermediate value theorem to show that the equation $x^3 + x = 1$ has a real solution. First state the Intermediate Value Theorem, then show precisely how the premise is satisfied and what conclusion follows.

   SOLN: The Intermediate Value Theorem states that if $f(x)$ is continuous on $[a, b]$ and n is between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = n$.

   In this case, let $f(x) = x^3 + x - 1$, so that $f(x) = 0 \Leftrightarrow x$ is a solution to the equation. Since f is a polynomial, f is continuous everywhere. In particular, it's continuous on $[0, 1]$ Now $f(0) = -1$ and $f(1) = 1$ So f, a continuous function, goes from negative to positive on $[0, 1]$ and by the IVT there exists $c \in (0, 1)$ such that $f(c) = 0$ and c is then a solution to the equation.

8. (10 points) Find a function f and a number a such that $\lim_{h\to 0} \frac{(2+h)^3 - 8}{h} = f'(a)$. ANS: Let $f(x) = x^3$, then the limit is $f'(2)$.

9. (10 points) The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



   SOLN: $f(x)$ is the curve labeled "a" works since then the curve labeled "c" is positive where f is increasing, 0 where f has a horizontal tangent line, and negative where f is decreasing. Similarly, the curve labeled "b" can be $f''(x)$ since it's positive where f' is increasing, has a zero where the line tangent to f' is horizontal and is negative where f' is decreasing.