Exercises from Stewart, 3.2:

$$f(x) = \frac{1+x}{1+e^x}$$
$$f'(x) = \frac{(1+e^x)1-(1+x)e^x}{(1+e^x)^2} = \frac{e^x(1-x)}{(1+e^x)^2}$$

$f'(0) = \frac{1}{4}$ so the equation of the tangent line at $\left(0, \frac{1}{2}\right)$ is

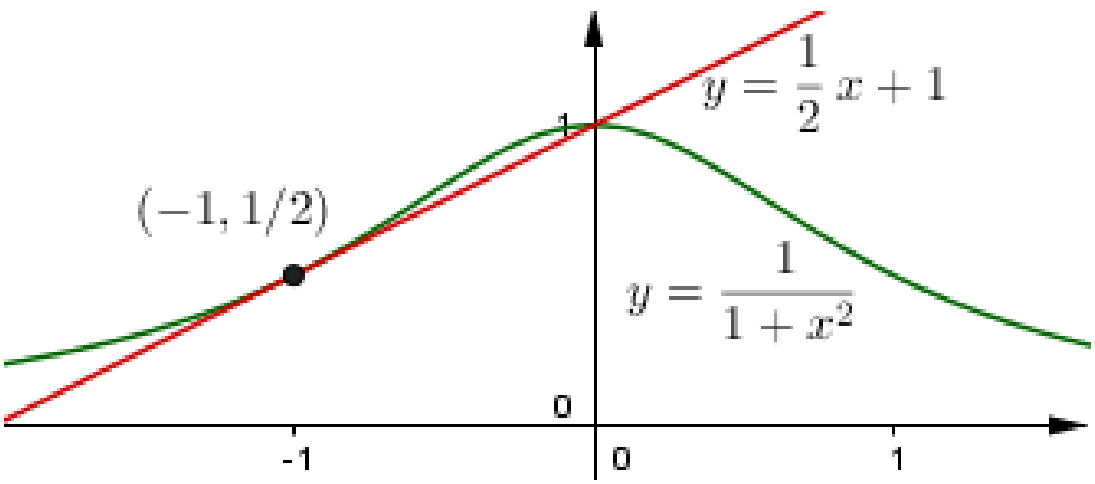
$$y - \frac{1}{2} = \frac{1}{4}(x-0) \Leftrightarrow y = \frac{1}{4}x + \frac{1}{2}$$

The "witch of Agnesi":

$$y = \frac{1}{1+x^2} \Rightarrow y' = \frac{dy}{dx} = -\frac{2x}{(1+x^2)^2}$$

The slope of the tangent line at $x = -1$ is $= -\frac{-2}{(1+1)^2} = \frac{1}{2}$ so the equation of the tangent

Line is $y - \frac{1}{2} = \frac{1}{2}(x+1) \Leftrightarrow y = \frac{1}{2}x + 1$



3.2 #42) If $g(x) = \frac{x}{e^x}$ find a formula for the *n*th derivative, $g^{(n)}(x)$.

Plan: Keep taking derivatives and look for a pattern.

$$g'(x) = \frac{e^x - xe^x}{e^{2x}} = \frac{1-x}{e^x} = \frac{1}{e^x} - g(x)$$
$$g''(x) = \frac{e^x(1-x)' - (1-x)(e^x)'}{e^{2x}}$$
$$= \frac{-e^x - (1-x)e^x}{e^{2x}} = \frac{-2+x}{e^x} = -\frac{1}{e^x} - g'(x)$$
$$-\frac{1}{e^x} - \left(\frac{1}{e^x} - g(x)\right) = -\frac{2}{e^x} + g(x)$$
$$g'''(x) = \frac{d}{dx}\left(\frac{-2+x}{e^x}\right) = \frac{d}{dx}\left(-\frac{2}{e^x} + g(x)\right)$$

$$= \frac{e^x(-2+x)' - (-2+x)(e^x)'}{e^{2x}} = \frac{e^x + 2e^x - xe^x}{e^{2x}} = \frac{3e^x - xe^x}{e^{2x}}$$

$$= \frac{3-x}{e^x} = \frac{3}{e^x} - g(x)$$

$$g'(x) = \frac{1}{e^x} - g(x)$$

$$g''(x) = -\frac{2}{e^x} + g(x)$$

$$g'''(x) = \frac{3}{e^x} - g(x)$$

Conjecture: $g^n(x) = (-1)^n \left(g(x) - \frac{n}{e^x}\right)$

Principle of Mathematical Induction Let S_n be a statement about the positive integer n. Suppose that

1. S_1 is true.

2. S_{k+1} is true whenever S_k is true.

Then S_n is true for all positive integers n.

Inductive hypothesis: $g^{(k)}(x) = (-1)^k \left(g(x) - \frac{k}{e^x}\right)$

then $g^{(k+1)}(x) = \frac{d}{dx} g^{(k)}(x) = (-1)^k \left(g'(x) + \frac{k}{e^x}\right)$

$$= (-1)^k \left(\frac{1}{e^x} - g(x) + \frac{k}{e^x}\right) = (-1)^k \left((-1)\left(g(x) - \frac{k+1}{e^x}\right)\right)$$

$= (-1)^{k+1} \left(g(x) - \frac{k+1}{e^x}\right)$

So we have established S_1 and shown that $S_k \Rightarrow S_{k+1}$