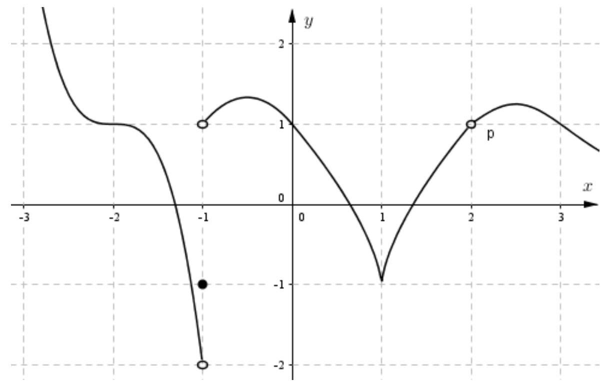
## Final Exam Solutions, Spring 2016

Write all responses on separate paper. Show your work in detail for credit. No calculators.

1. (12 points) Consider the function defined by its schematic graph below.




   (a) Find each limit or write "DNE" if the limit does not exist.

   i $\lim_{x \to -1^-} f(x) = -2$ ii $\lim_{x \to -1^+} f(x) = 1$ iii $\lim_{x \to 2} f(x) = 1$

   (b) Find all the discontinuities and classify each as either a removable discontinuity, a jump discontinuity or a vertical asymptote. ANS: There is a jump discontinuity at $x = -1$ and a removable discontinuity where $x = 2$.

   (c) Approximate $x$ for all points where $f'(x) = 0$.
   ANS: The slope of the tangent line is horizontal where $x = -2, -1/2$, and $x = 2.5$.

   (d) Approximate $x$ for all points where $f'(x) = -\frac{1}{2}$ (approximate to the nearest tenth.)
   ANS: The solope of the tangent line is $-\frac{1}{2}$ where $x \approx -2.3, -1.7, -0.3$, and 2.7

2. (12 points) Use the **definition** of the derivative (that is, $f'(x) \equiv \lim_{a \to x} \frac{f(x) - f(a)}{x - a}$) to show that $\frac{d}{dx}\left(\frac{2}{x}\right) = -\frac{2}{x^2}$.

   SOLN: $\lim_{a \to x} \frac{\frac{2}{x} - \frac{2}{a}}{x - a} = \lim_{a \to x} \frac{2a - 2x}{ax(x-a)} = \lim_{a \to x} \frac{-2(x-a)}{ax(x-a)} = \lim_{a \to x} \frac{-2}{ax} = -\frac{2}{x^2}$

3. (12 points) The cost of living adjustment (COLA) is used by the government to ensure that the purchasing power of benefits is not diminished by inflation of prices. Let $t$ = years since 2014 and $C(t)$ = COLA in the year $t$. Values of $C$ are tabulated below

| $t$ | 0 | 1 | 2 |
|---|---|---|---|
| $C(t)$ | 1.5% | 1.7% | 0 |

   (a) Approximate $C'(1)$ as the average rate of change for $0 \le t \le 2$.
   ANS: $C'(1) \approx \frac{0 - 1.5}{2} = -0.75$

   (b) Find coefficients $a, b$, and $c$ so that $C(t) = at^2 + bt + c$ agrees with the points in the table. Then use this quadratic function to approximate $C'(1)$.
   ANS: $C(t) = at^2 + bt + 1.5$ so $C(1) = a + b + 1.5 = 1.7$ and $C(2) = 4a + 2b + 1.5 = 0$. Solving the first for $a = 0.2 - b$ and substituting into the second, $4(0.2 - b) + 2b = -1.5 \Leftrightarrow 2b = 2.3 \Leftrightarrow b = 1.15$ so $C(t) = -0.95t^2 + 1.15t + 1.5$ whence $C'(1) = -1.9 + 1.15 = -0.75$ Hey, same answer!

   (c) Describe in words what the meaning of $C'(1)$ is.
   $C'(1)$ is the instantaneous rate of change in the COLA measured in percentage points per year.

4. (12 points) Let $f(x) = \arctan(x)$ on $[1, \sqrt{3}]$.

(a) Explain why the function satisfies the conditions of the Mean Value Theorem.
$f$ is differentiable and continuous everywhere.

(b) Find all values of $c$ which satisfy the conclusion of the Mean Value Theorem.
$f'(x) = \frac{1}{1+c^2} = \frac{\arctan(\sqrt{3}) - \arctan(1)}{\sqrt{3}-1} = \frac{\frac{\pi}{3}-\frac{\pi}{4}}{\sqrt{3}-1} = \frac{\pi}{12(\sqrt{3}-1)} \Leftrightarrow c^2+1 = \frac{12(\sqrt{3}-1)}{\pi}$
$\Rightarrow c = \sqrt{\frac{12(\sqrt{3}-1)-\pi}{\pi}}$

5. (12 points) Consider the function

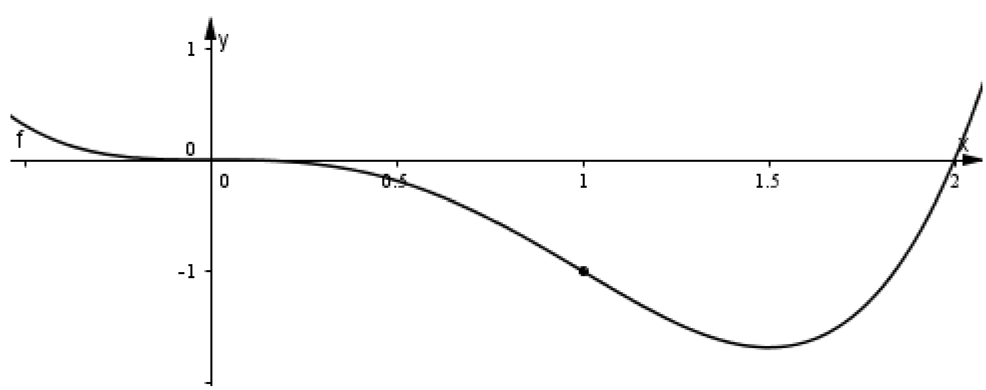
$$f(x) = \begin{cases} \frac{7-16^{\frac{1}{x}}}{1+16^{\frac{1}{x}}} & \text{if} \quad x \neq 0 \\ 7 & \text{if} \quad x = 0 \end{cases}$$

(a) Is $f(x)$ continuous at $x = 0$? *Hint:* $\frac{d}{dx}b^x = \ln b \cdot b^x$
We need to check that the left and right hand limits both exist and agree. First, $\lim_{x\to 0^-} f(x) = 7$, since the exponents are going to $-\infty$ and $\lim_{x\to-\infty} 16^x = 0$. Next, $\lim_{x\to 0^+} f(x) = -1$. That's not so obvious unless you notes that both numerator and denominator go to $\infty$ as $x \to 0^+$ and use L'Hospital's rule: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\frac{1}{x^2}\ln 16 \cdot 16^{\frac{1}{x}}}{-\frac{1}{x^2}\ln 16 \cdot 16^{\frac{1}{x}}} = -1$, so the left and right limits don't agree and the function is discontinuous at $x = 0$.

(b) Prove that the equation $f(x) = 1$ has a solution in the interval $[2, 4]$.
ANS: $f$ is continuous on $[2, 4]$ and $f(2) = \frac{3}{5}$ while $f(4) = \frac{5}{3}$. Since 1 is between $f(2)$ and $f(4)$, by the Intermediate Value Theorem, there exists $n \in (2, 4)$ such that $f(n) = 1$.

6. (8 points) An equation of the form $p(t) = Ae^{-ct}\sin(\omega t + \delta)$ represents the position of a object at time $t$. Find the velocity and acceleration of the object.
ANS: Velocity $= v(t) = p'(t) = -cAe^{-ct}\sin(\omega t+\delta) + \omega Ae^{-ct}\cos(\omega t+\delta) = Ae^{-ct}(\omega\cos(\omega t+\delta) - c\sin(\omega t+\delta))$
$= \sqrt{\omega^2+c^2}Ae^{-ct}\sin\left(\omega t+\delta - \arctan\left(\frac{\omega}{c}\right)\right)$
Acceleration $= a(t) = v'(t) = c^2Ae^{-ct}\sin(\omega t+\delta) - 2c\omega Ae^{-ct}\cos(\omega t+\delta) - \omega^2Ae^{-ct}\sin(\omega t+\delta)$
$= e^{-ct}A((c^2-\omega^2)\sin(\omega t+\delta) - 2c\omega A\cos(\omega t+\delta)) = Ae^{-ct}(c^2+\omega^2)\sin\left(\omega t+\delta - \arctan\left(\frac{2c\omega}{c^2-\omega^2}\right)\right)$

7. (8 points) For what values of $a$ and $b$ is $(1, -1)$ and inflection point on the curve $f(x) = ax^4 + bx^3$?
ANS: First, require that $f(1) = a + b = -1$ Then $f''(1) = 12a + 6b = 0 \Leftrightarrow b = -2a$ and substitute back: $a - 2a = -1 \Leftrightarrow a = 1$ so $b = -2$ and the curve is $f(x) = x^4 - 2x^3$. To be sure, here's a graph:

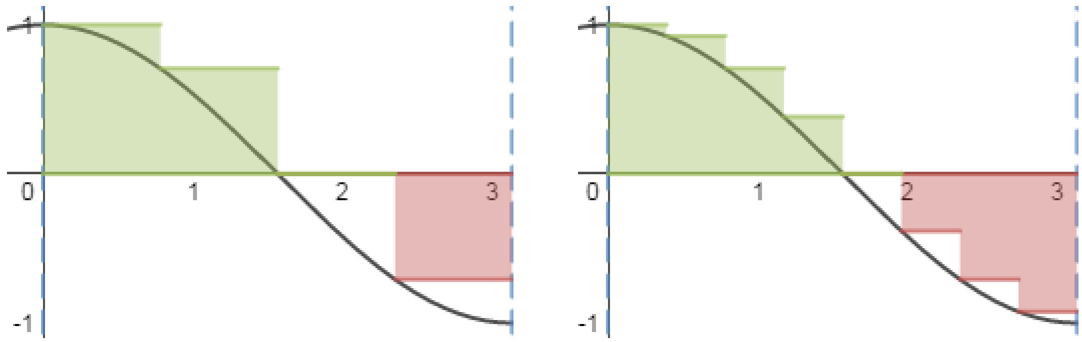



8. (8 points) Evaluate the upper and lower approximating sums for $\int_0^{\pi} \cos(x)dx$ with $n = 4$ and $n = 8$.
Since the integrand is a decreasing function on $[0, \pi]$ the upper sums will be

$$L_4 = \frac{\pi}{4}\left(\cos(0) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}\left(1 + \frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

and

$L_8 = \frac{\pi}{8}\left(\cos(0) + \cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{8}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{5\pi}{8}\right) + \cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{7\pi}{8}\right)\right) = \frac{\pi}{8}$ In the last computation the symmetry pattern was used: observing that each function value will be paired with an opposite, except for the first. To see this, draw some diagrams:

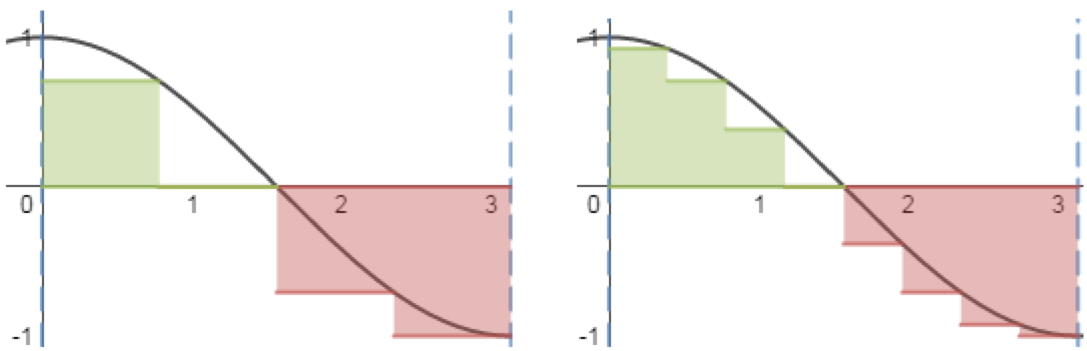



Similarly the lower sums are

$$R_4 = \frac{\pi}{4}\left(\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{3\pi}{4}\right) \cos(\pi)\right) = \frac{\pi}{4}\left(\frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{2}}{2} - 1\right) = -\frac{\pi}{4}$$

and

$$R_8 = \frac{\pi}{8}\left(\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{8}\right) + \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{5\pi}{8}\right) + \cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{7\pi}{8}\right) + \cos(\pi)\right) = -\frac{\pi}{8}$$




9. (8 points) Consider the integral function $g(x) = \int_0^x (1-t^2)e^{t^2}\,dt$

   (a) Where is $g(x)$ increasing?
   ANS: $g'(x) = (1-x^2)e^{x^2} > 0$ on $(-\infty,-1)\cup(1,\infty)$

   (b) Find $g$'s inflection point.
   ANS: $g''(x) = -2x^3e^{x^2}$ changes sign where $x = 0$.

10. (8 points) Derive the formula for Newton's method and use it to find $x_2$ if $x_1 = \frac{\pi}{2}$ and we're searching for a zero of $f(x) = x - 2\sin x$.
ANS: The equation for the tangent line to $y = f(x)$ at $x_n$ is $y - f(x_n) = f'(x_n)(x - x_n)$ The $x-$intercept of that line is by plugging in $y = 0$ and solving for $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. In the case of $f(x) = x - 2\sin x$ with $x_1 = \frac{\pi}{2}$,

$$x_2 = \frac{\pi}{2} - \frac{\frac{\pi}{2} - 2\sin\left(\frac{\pi}{2}\right)}{1 - 2\cos\left(\frac{\pi}{2}\right)} = 2$$