Math 1A **Name (Print):** ____________________

Chapter 2 Test

Write all responses on separate paper. Show your work in detail for credit. No calculators.

1. Let $P(t)$ be the inches of precipitation at Horse Meadow on day t where $t =$ the number of days since 12/27/2005. The table at right shows the value of this fucntion over a 5 day period.

Date	$P(t)$ (in.)
12/27/2005	0.7
12/28/2005	0.5
12/29/2005	0
12/30/2005	0.9
12/31/2005	1.6

   (a) Use the table to find the average rate of change in precipitation between 12/27/2005 and 12/31/2005. Be sure to specify the units of measure for this rate of change.

   Soln: $\dfrac{\Delta P}{\Delta t} = \dfrac{1.6 - 0.7}{4 - 0} = \dfrac{0.9}{4} = 0.225 \approx 0.2 \dfrac{\textbf{inch}}{\textbf{day}}$

   (b) What is your best approximation, based on this table, for rate of change on December 29?

   It's probably best to use data nearer to the point, but not biased to one side or the other. It turns out if you average the slopes before and after you get the same as the slope of the segment connecting $(1, P(1))$ to $(3, P(3))$:

$$\frac{\frac{P(2)-P(1)}{2-1} + \frac{P(3)-P(2)}{3-2}}{2} = \frac{P(3)-P(1)}{3-1} = \frac{0.9-0.5}{2} = 0.2\frac{\textbf{inch}}{\textbf{day}}$$

2. Find the limit. Give baby steps and justify each step with a property of limits.

   (a) $\displaystyle\lim_{x\to 2} \frac{x^3-8}{x^3-x^2-x-2}$.

   SOLN: $\displaystyle\lim_{x\to 2} \frac{x^3-8}{x^3-x^2-x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x^2+x+1)} = \lim_{x\to 2} \frac{x^2+2x+4}{x^2+x+1} = \frac{12}{7}$

   (b) $\displaystyle\lim_{x\to 2} \frac{4-x^2}{|4-x^2|}$

   SOLN: One way to work this is to first write the absolute value as a piece-wise defined function:

$$|4-x^2| = \begin{cases} x^2-4 & : x \le -2 \\ 4-x^2 & : -2 \le x \le 2 \\ x^2-4 & : x > 2 \end{cases}$$

   Thus $\displaystyle\lim_{x\to 2^-} \frac{4-x^2}{|4-x^2|} = \lim_{x\to 2^-} \frac{4-x^2}{4-x^2} = 1$, but $\displaystyle\lim_{x\to 2^+} \frac{4-x^2}{|4-x^2|} = \lim_{x\to 2^-} \frac{4-x^2}{x^2-4} = -1$, so the limit does not exist.

   (c) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2-100}}{3x-4}$

   SOLN: $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2-100}}{3x-4} = \lim_{x\to\infty} \frac{\sqrt{x^2-100}}{3x-4}\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{1-100/x^2}}{3-4/x} = \frac{\sqrt{1-\lim_{x\to\infty} 100/x^2}}{3-\lim_{x\to\infty} 4/x} = \frac{1}{3}$

   (d) $\displaystyle\lim_{x\to\pi^-} \ln\left(\tan\left(\tfrac{x}{2}\right)\right)$

   SOLN: $\displaystyle\lim_{x\to\pi^-} \ln\left(\tan\left(\tfrac{x}{2}\right)\right) = \ln\left(\lim_{x\to\pi^-} \tan\left(\tfrac{x}{2}\right)\right) = \ln(\infty) = \infty$

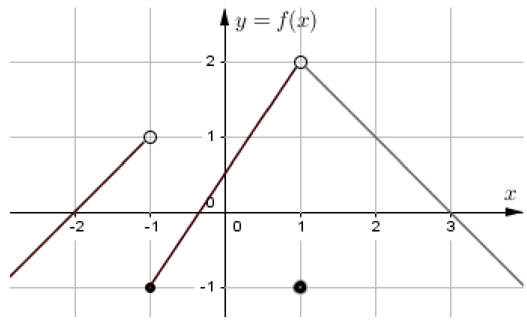
3. Give a formal epsilon-delta definition proof that $\displaystyle\lim_{x\to 3} 2x+1 = 7$

   First we work backwards from the conclusion we need to arrive at to see what premise might ensure that conclusion. The conclusion is that $|f(x)-L| = |2x+1-7| = |2x-6| = 2|x-3| < \epsilon \Leftrightarrow |x-3| < \epsilon/2$. So choose $\delta = \epsilon/2$ and the conclusion follows by following the equivalence chain backwards: Let $\delta = \epsilon/2$ then $|x-a| = |x-3| < \delta \Leftrightarrow |x-3| < \epsilon/2 \Leftrightarrow 2|x-3| < \epsilon \Leftrightarrow |2x+1-7| < \epsilon \Leftrightarrow |f(x)-L| < \epsilon$.

4. Find the smallest value of N so that if $x > N$ then $\frac{\pi}{2} - \arctan(x) < \frac{\pi}{4}$
SOLN: $\frac{\pi}{2} - \arctan(x) < \frac{\pi}{4} \Leftrightarrow \arctan(x) > \frac{\pi}{4} \Leftrightarrow x > 1$. Choose $N = 1$.

5. Consider the graph of the function $y = f(x)$ shown below:
   (a) Find each limit, or explain why it does not exist. SOLNS:

   i. $\lim_{x\to-1^-} f(x) = 1$ ii. $\lim_{x\to-1^+} f(x) - 1$ iii. $\lim_{x\to1} f(x) = 2$



   (b) Is f discontinuous where $x = -1$? Justify your answer using the definition of continuity.
   SOLN: Yes, f is discontinuous where $x = -1$ since the limit from the left is not the same as the limit from the right. This is a **jump discontinuity.**

   (c) Is f discontinuous where $x = 1$? Justify your answer using the definition of continuity.
   SOLN: Yes, f has a **removable discontinuity** where $x = 1$ since, while the limit exists, the value of the function at $x = 1$ is not the same as the limit.

6. Let

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & : x \le 2 \\ 1+b & : x = 2 \\ (x-2)^3 + 4 & : x > 2 \end{cases}$$

Find all values of b so that f is a continuous function.
SOLN: First observe that $\lim_{x\to2^-} f(x) = 4$ and $\lim_{x\to2^+} f(x) = \lim_{x\to2^+} f(x) = 4$. Thus continuity requires that $f(2) = 1 + b - 4$. So $b = 3$.

7. Consider $f(x) = x^2 + x$
   (a) Use the **definition** of the derivative to find the derivative function $f'(x)$
   SOLN: $f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to0} \frac{(x+h)^2 + (x+h) - x^2 - x}{h}$
   $= \lim_{h\to0} \frac{x^2 + 2hx + h^2 + x + h - x^2 - x}{h} = \lim_{h\to0} \frac{2hx + h^2 + h}{h} = \lim_{h\to0} 2x + h + 1 = 2x + 1$
   (b) Find an equation for the line tangent to the function where $x = 1$
   SOLN: The slope is $f'(1) = 3$ and the point slope formula for a line yields $y - 2 = 3(x-1) \Leftrightarrow y = 3x - 1$

8. Use the intermediate value theorem to show that the equation

$$\sin(\pi\sqrt{x}) = 4x^2 - 4x + 1$$

has a solution for $0 < x < 1$. First state the Intermediate Value Theorem, then show precisely how the premise is satisfied and what conclusion follows. SOLN: The Intermediate Value Theorem states that if $f(x)$ is continuous on $[a, b]$ and N is between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = N$.
In this case, let $f(x) = 4x^2 - 4x + 1 - \sin(\pi\sqrt{x})$. As a difference of continuous functions, this function is continuous. In particular, it's continuous on $[0, 1]$. A solution, $x = a$, to the original equation is a value where $f(a) = 0$. Now $f(0) = 1$ and $f(1) = 1$. Hmmm...but there are no other numbers between 1 and 1! So we look at values in between. A convenient value is $a = \frac{1}{4}$ where $f\left(\frac{1}{4}\right) = 4\left(\frac{1}{4}\right)^2 - 4\left(\frac{1}{4}\right) + 1 - \sin\left(\pi\sqrt{\frac{1}{4}}\right) = \frac{1}{4} - 1 = -\frac{3}{4} < 0$. Thus, f, a continuous function, goes from positive to negative on $[0, 1/4]$ and then from negative to positive on $[1/4, 1]$, so by the IVT there exist two values $c \in [0, 1]$ such that $f(c) = 0$ and both of these c are then solutions to the equation.