Math 1A–Spring 16–Chapter 4 Test Solns

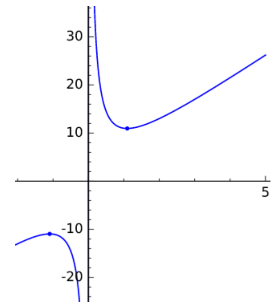
1. (12 points) For each function, find the critical points and the intervals of increase and decrease.

   (a) $f(x) = 5x + 6x^{-1}$

   SOLN: $f'(x) = 5 - 6/x^2 = 0 \Leftrightarrow x = \pm\frac{\sqrt{30}}{5}$. The function follows the slant asymptote $y = 5x$ for $|y| >> 0$ and the vertical asymptote along $x = 0$ for small x.

   f increases on $\left(-\infty, -\frac{\sqrt{30}}{5}\right) \cup \left(\frac{\sqrt{30}}{5}, \infty\right)$

   and is decreasing on $\left(-\frac{\sqrt{30}}{5}, 0\right) \cup \left(0, \frac{\sqrt{30}}{5}\right)$.



   (b) $h(x) = \frac{x}{(3x^2-5)^{1/3}}$

   SOLN: $h'(x) = (3x^2-5)^{-1/3} - 2x^2(3x^2-5)^{-4/3}$
   $= (3x^2-5)^{-4/3}(3x^2-5-2x^2) = (3x^2-5)^{-4/3}(x^2-5)$
   So $h'(x) = 0 \Leftrightarrow x = \pm\sqrt{5}$. A little examination shows the function is increasing on $(-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty)$ and decreasing on $\left(-\sqrt{5}, -\frac{\sqrt{15}}{3}\right) \cup \left(-\frac{\sqrt{15}}{3}, \frac{\sqrt{15}}{3}\right) \cup \left(\frac{\sqrt{15}}{3}, \sqrt{5}\right)$

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f(x)=x/(sign(3*x^2-5)*abs(3*x^2-5)^(1/3))
p=plot(f,(x,-5,5))
p+=point([-sqrt(5),-sqrt(5)/(10)^(1/3)])
p+=point([sqrt(5),sqrt(5)/(10)^(1/3)])
p.show(xmin=-5, xmax=5, ymin=-2, ymax=2)
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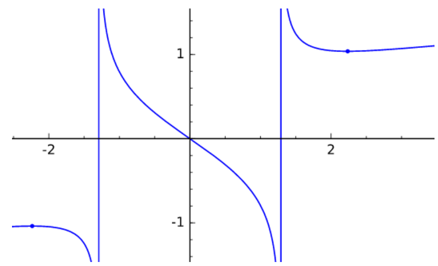
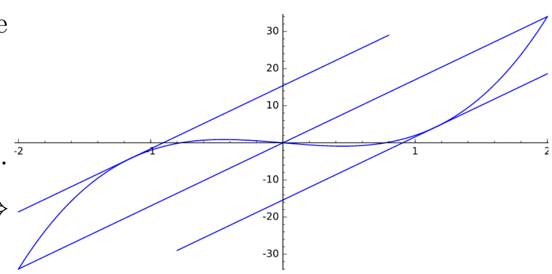


2. (12 points) Let $f(x) = 5x^3 - 3x$ on the interval $[-2, 2]$.

   (a) State why this function satisfies the conditions of the Mean Value Theorem.
   f is continuous on $[-2, 2]$ and differentiable on the interior.

   (b) Find all values $c \in (-2, 2)$ which satisfy the conclusion of the MVT.
   $f'(c) = 15c^2 - 3 = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{34 - (-34)}{4} = 17 \Leftrightarrow c^2 = \frac{4}{3} \Leftrightarrow$
   $c = \pm\frac{2\sqrt{3}}{3} \approx \pm 1.15 \in (-2, 2)$



3. (10 points) Suppose that $f(x)$ is differentiable for all x and that $2 \le f'(x) \le 6$ for all values of x. Show that $4 \le f(5) - f(3) \le 12$.
   SOLN: By the MVT, every secant line has a parallel tangent line, so the slope of a secant line is also bound by 2 and 6. On the interval $(3, 5)$, in particular, $2 \le \frac{f(5) - f(3)}{5 - 3} \le 6 \Leftrightarrow 4 \le f(5) - f(3) \le 12$, as desired.

4. (20 points) Suppose $f(x) = \frac{20x}{x^2 + 25}$

   (a) What is the domain of f? SOLN: Domain $= \mathbb{R}$

   (b) What intercept(s) does f have? SOLN: The only intercept is $(0, 0)$.

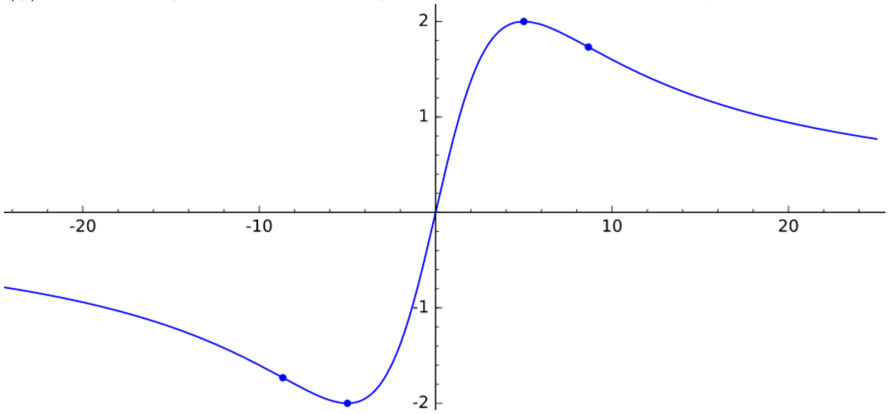
   (c) Describe the symmetry of f. SLN: f has odd symmetry.

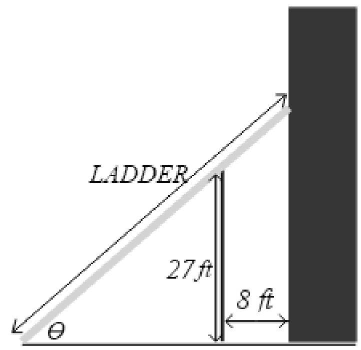
   (d) What asymptote(s) does f have? SOLN: $y = 0$ is a horizontal asymptote.

   (e) Simplify $f'(x)$ and use it to find critical numbers and intervals of increase/decrease.
   $f'(x) = \frac{20(x^2+25) - 20x(2x}{(x^2+25)^2} = \frac{500 - 20x^2}{(x^2+25)^2} = 0 \Leftrightarrow x = \pm 5$ so f is decreasing on $(\infty, -5) \cup (5, infty)$ and increasing on $(-5, 5)$

   (f) Simplify $f''(x)$ and use that to find inflection points of f. $f''(x) = \frac{-40x(x^2+25)^2 - 80x(x^2+25)(25-x^2)}{(x^2+25)^4} =$
   $\frac{-40x(x^2+25) - 80x(25-x^2)}{(x^2+25)^3} = \frac{40x(x^2-75)}{(x^2+25)^3} = 0 \Leftrightarrow x = 0$ or $x = \pm 5\sqrt{3} \approx \pm 8.66$

(g) Sketch a graph illustrating all these components of f.





5. (16 points) A wall 27 feet tall runs parallel to a tall building at a distance of 8 ft from the building as shown in the diagram (not to scale). We wish to find the length of the shortest ladder that will reach from the ground over the fence to the wall of the building.

   (a) First, find a formula for the length of the ladder, $L(\theta)$, in terms of θ. (*Hint*: split the ladder into 2 parts.)

   (b) To find the min. length of the ladder, solve $L'(\theta) = 0$ and then use the critical value of θ_m to simplify $L(\theta_m)$.

SOLN: Split the ladder in two length so that $L = a + b = \dfrac{27}{\sin\theta} + \dfrac{8}{\cos\theta}$. Then $L'(\theta) = \dfrac{-27\cos\theta}{\sin^2\theta} + \dfrac{8\sin\theta}{\cos^2\theta} = 0 \Leftrightarrow$ $8\sin^3\theta = 27\cos^3\theta \Leftrightarrow \theta = \arctan\left(\frac{3}{2}\right)$ so that $L\left(\arctan\left(\frac{3}{2}\right)\right) = 9\sqrt{139} + 4\sqrt{13} = 13\sqrt{13} \approx 47$ft.

6. (16 points) Answer the following questions about approximation with Newton's method:

   (a) Starting with an initial value of 3, use two iterations of Newton's method to approximate a zero of $f(x) = x^3 - 4x^2 - 2x + 14$. Simplify.
   SOLN: $x_(n+1) = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 - 4x_n^2 - 2x_n + 14}{3x_n^2 - 8x_n - 2} = \dfrac{2x_n^2(x_n-2) - 14}{3x_n^2 - 8x_n - 2} = dfrac18 - 141 = 4$. What luck, an integer! Then $x_2 = \dfrac{64-14}{48-32-2} = \dfrac{25}{7}$.
   In Sagemath, you can use Python to do this like so:

```
x = var('x');
f(x)=x^3-4*x^2-2*x+14;
df=diff(f,x);
NewtonIt(x)=x-(f/df)(x);

xn=3;
print xn;
for i in range(20):
    xn=N(NewtonIt(xn),digits=20)
    print xn;
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3
4.0000000000000000000
3.5714285714285714286
3.3906782872300113679
3.3508892803805082401
3.3488991543703890906
3.3488942175310772078
3.3488942175007169798
3.3488942175007169798
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   (b) Choose $x_0 = 4$ to be an initial approximation of $\sqrt{13}$. Use one step of Newton's method on an appropriately chosen polynomial function to develop x_1, a better rational approximation of $\sqrt{13}$; also give an arithmetic expression for the better approximation x_2 arising from a second step of Newton's method.
   SOLN: The simplest function with a zero at $\sqrt{13}$ is $f(x) = x^2 - 13$, the iterates of this are given by the well-known Babylonian formula: $x_{n+1} = \dfrac{x_n + 13/x_n}{2}$ With $x_0 = 4$ we have $x_1 = \dfrac{4 + 13/4}{2} = \dfrac{16+13}{8} = \dfrac{29}{8}$ and $x_2 = \dfrac{\frac{29}{8} + \frac{104}{29}}{2} = \dfrac{29^2 + 832}{16 \cdot 29} = \dfrac{1673}{464} \approx 3.606$

7. (14 points) Compute the limit:

   (a) $\displaystyle\lim_{t\to 0} \frac{t - \sin t}{t^3} = \lim_{t\to 0} \frac{1 - \cos t}{3t^2} = \lim_{t\to 0} \frac{\sin t}{6t} = \frac{1}{6}$

   (b) $\displaystyle\lim_{x\to\infty} xe^{1/x} - x = \lim_{x\to\infty} \frac{e^{1/x} - 1}{1/x} = \lim_{x\to\infty} \frac{-e^{1/x}/x^2}{-1/x^2} = \lim_{x\to\infty} e^{1/x} = e^0 = 1$