Math 1A **Name (Print):** ____________________

Chapter 3 Test-1

Write all responses on separate paper. Show your work in detail for credit. No calculators.

1. (18 points) Compute

   (a) $\lim_{x\to 0} \frac{(1+x)^{1/3}-(1+\frac{1}{3}x)}{x^2}$
   SOLN: The numerator and denominator both approach zero, so L'Hopital's rule applies:
   $\lim_{x\to 0} \frac{(1+x)^{1/3}-(1+\frac{1}{3}x)}{x^2} = \lim_{x\to 0} \frac{\frac{1}{3}(1+x)^{-2/3}-\frac{1}{3}}{2x}$. The numerator and denominator are still both approaching zero, so we need to use L'Hoptial's rule again: $\lim_{x\to 0} \frac{\frac{-2}{9}(1+x)^{-5/3}}{2} = -\frac{1}{9}$

   (b) $\lim_{n\to\infty} \frac{n^3+3n^2}{2n^3-n}$ SOLN: The numerator and denominator are both going to infinity, so, again, L'Hopital's rule applies: $\lim_{n\to\infty} \frac{n^3+3n^2}{2n^3-n} = \lim_{n\to\infty} \frac{3n^2+6n}{6n^2-1} = \lim_{n\to\infty} \frac{6n+6}{12n} = \lim_{n\to\infty} \frac{6}{12} = \frac{1}{2}$. Of course, you could also divide by $\frac{x^3}{x^3}$ which might be simpler.

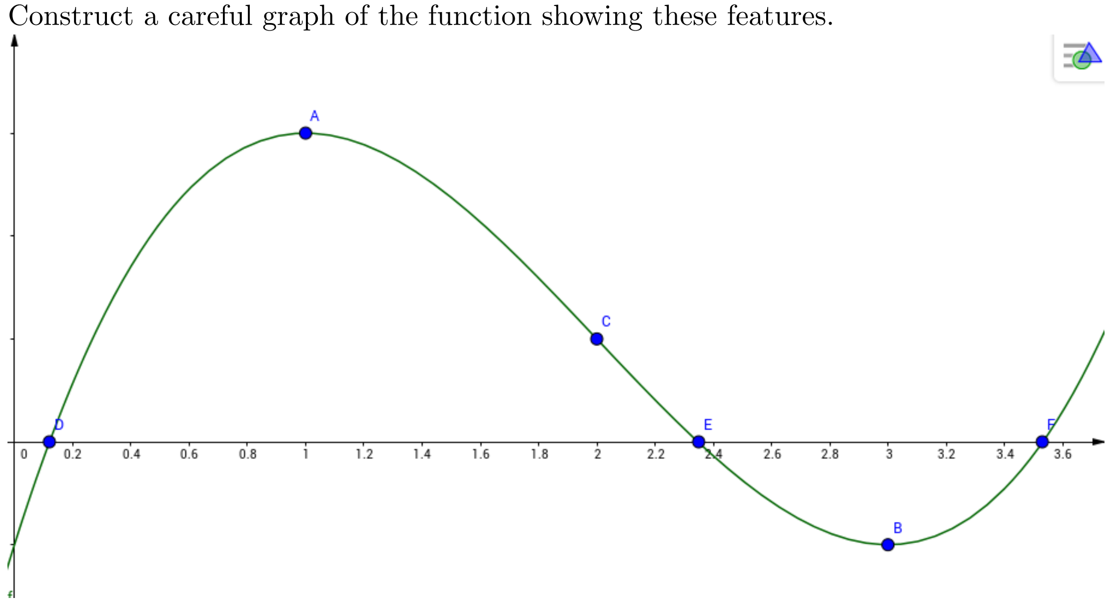
   (c) $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x}$
   SOLN: $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0^+} \frac{\sin x - x}{x\sin x} = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} = \lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x\sin x} = 0$, which makes sense, since $\sin x \to x$ as $x \to 0$

2. (15 points) Let $f(x) = 2x^3 - 12x^2 + 18x - 2$ be a cubic polynomial with zeros $x_1 \approx 0.12, x_2 \approx 2.35, x_3 \approx 3.53$

   (a) Find the critical points of the function.
   SOLN: Since the function is differentiable everywhere, the only critical points are where $f'(x) = 6x^2 - 24x + 18 = 0 \Leftrightarrow (x-3)(x-1) = 0 \Leftrightarrow x = 3$ or $x = 1$. So the critical points are are $(1, 6)$ and $(3, -2)$.

   (b) Find the inflection point.
   By the symmetry of a cubic polynomial, the inflection point occurs halfway between the critical points, at $(2, 2)$. OF course you can also compute the second derivative and see where it changes sign. $f"(x) = 12x - 24$ changes sign at 2.

   (c) Construct a careful graph of the function showing these features.



3. (15 points) Let $f(\theta) = 2\sin\theta + \sin 2\theta$ on $\theta \in [0, 2\pi]$

   (a) Find the intercepts of the function.
   $2\sin\theta + \sin 2\theta = 0 \Leftrightarrow 2\sin\theta + 2\sin\theta\cos\theta = 0 \Leftrightarrow 2\sin\theta(1+\cos\theta) = 0$, so either $\sin\theta = 0$ wheich means $\theta = 0$ or $\theta = \pi$ or $\theta = 2\pi$, or $\cos\theta = -1$ which produces $\theta = \pi$ again. The intercepts are $(0,0), (\pi, 0)$ and $(2\pi, 0)$.

   (b) Find the extreme values of the function. The function is differentiable everywhere, so the only critical numbers are where $f'(\theta) = 0 \Leftrightarrow 2\cos\theta + 2\cos 2\theta = 0 \Leftrightarrow 2\cos^2\theta + \cos\theta - 1 = 0 \Leftrightarrow\Leftrightarrow \left(\cos\theta + \frac{1}{4}\right)^2 = \frac{1}{2} + \frac{1}{16} \Leftrightarrow$ $\cos\theta = -\frac{1}{4} \pm \frac{3}{4}$ so the critical numbers are where $\cos\theta = -1 \Leftrightarrow \theta = \pi$ and where $\cos\theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}, \frac{4\pi}{3}$. We can use the second derivative test to classify these as either max or min (or neither.) The second derivative is $f''(\theta) = -2\sin\theta - 4\sin(2\theta)$ and this gives $f''\left(\frac{\pi}{3}\right) = -\sqrt{3} - 4\sqrt{3} = -6\sqrt{3} < 0$ indicating that $f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a maximum. Similarly, $f\left(\frac{4\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a minimum. But $f''(\pi) = 0$ is inconclusive. However, in a neighborhood of $\theta = \pi$, $f'(\theta) = 2\cos\theta + 2\cos 2\theta = 4\cos^2\theta + 2\cos\theta - 2 = 2\left(\cos\theta + \frac{1}{4}\right)^2 - \frac{9}{8} \leq 0$, so the function is decreasing before and after $\theta = \pi$, and thus $f(\pi) = 0$ is neither a max nor a min.

   (c) Find the inflection points.
   $f''(\theta) = -2\sin\theta - 4\sin(2\theta) = -2\sin\theta - 8\sin\theta\cos\theta = -2\sin\theta(1 + 4\cos\theta)$ changes sign at $(\pi, 0)$ and where $\cos\theta = -\frac{1}{4} \Leftrightarrow \theta = \arccos\left(-\frac{1}{4}\right) = \pi - \arccos\left(\frac{1}{4}\right)$ or $\theta = 2\pi - \arccos\left(-\frac{1}{4}\right) = \pi + \arccos\left(\frac{1}{4}\right)$
   To be sure, we could enter these commands into sagemath:

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f=2*sin(x)+sin(2*x)
P1=plot(f,(x,0,6.28))
P1+=points((pi/3,1.5*sqrt(3)))+points((5*pi/3,-1.5*sqrt(3)))+points((pi,0))+
points((pi-arccos(0.25),f(pi-arccos(0.25))))+points((pi+arccos(0.25),f(pi+arccos(0.25))))
show(P1)
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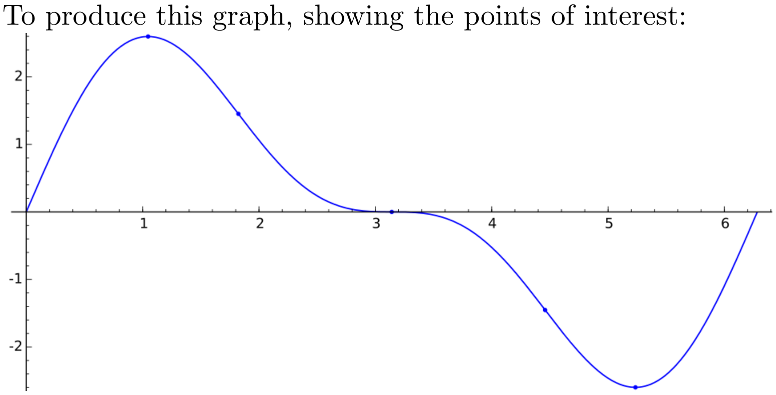
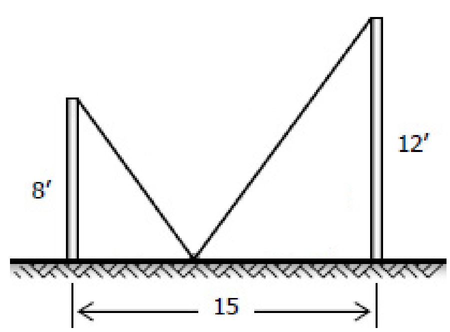
   To produce this graph, showing the points of interest:



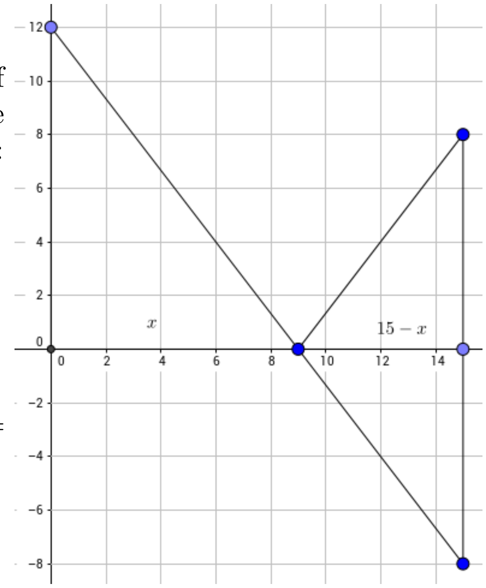
4. (19 points) The figure at right shows two vertical posts, 8 and 12 feet high, which stand 15 feet apart. A wire is to join the tops of these posts and a stake on the ground between them. Use calculus to determine where to place the stake to minimize the length of wire needed?



Start by introducing a control variable.

SOLN: Let x = distance from one post to the stake so that $15 - x$ is the distance from the other post to the stake. By the Pythagorean theorem, the sum of the lengths is $f(x) = \sqrt{x^2 + 12^2} + \sqrt{(15-x)^2 + 8^2}$. To minimize, solve $f'(x) = 0 \Leftrightarrow \frac{x}{\sqrt{x^2+12^2}} = \frac{15-x}{\sqrt{(15-x)^2+8^2}} \Leftrightarrow x^2[(15-x)^2 + 8^2] = (15-x)^2(x^2+12^2) \Leftrightarrow x^2(15-x)^2 + 8^2x^2 = (15-x)^2x^2 + 12^2(15-x)^2 \Leftrightarrow (12^2 - 8^2)x^2 - 2\cdot 15\cdot 12^2 x + 12^2 15^2 \Leftrightarrow ((12-8)x - 12\cdot 15)((12+8)x - 12\cdot 15) \Leftrightarrow$ $80(x-45)(x-9) = 0$. Thus, $x = 9$ feet.

Of course, there is an easier solution using geometry instead of calculus. Reflect one pole across the line of the ground and note that the shortest line connecting the two points is a straight line:



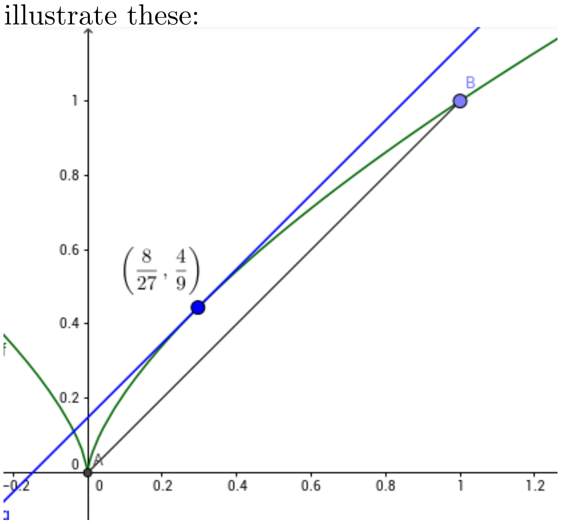
5. (18 points) Find a number $c \in (A, B)$ such that $f'(c) = \dfrac{f(B) - f(A)}{B - A}$ where

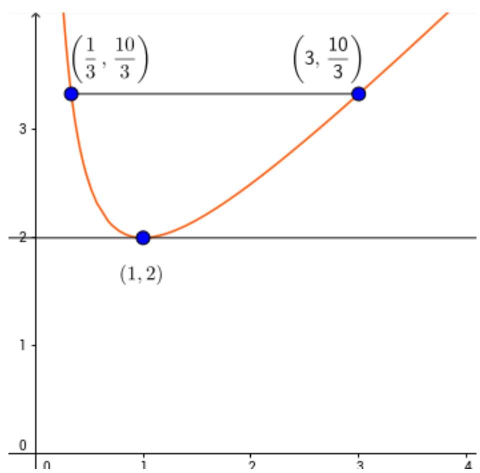
(a) $f(x) = x^{2/3}$ and $A = 0, B = 1$

SOLN: We seek $c \in (0, 1)$ such that $f'(c) = \dfrac{f(1) - f(0)}{1 - 0} = 1 \Leftrightarrow \frac{2}{3}c^{-1/3} = 1 \Leftrightarrow c = \dfrac{8}{27}$

(b) $f(x) = x + \dfrac{1}{x}$ and $A = \frac{1}{3}, B = 3$

SOLN: We seek $c \in \left(\dfrac{1}{3}, 3\right)$ such that $f'(c) = \dfrac{f(3) - f\left(\frac{1}{3}\right)}{3 - \frac{1}{3}} = 0 \Leftrightarrow 1 - \dfrac{1}{c^2} = 0 \Rightarrow c = 1$. Here're pictures to illustrate these:





6. (15 points) We will use Newton's method to find $\sqrt[3]{A}$

(a) Show $x = \sqrt[3]{A}$ is a root of $f(x) = x^3 - A$

SOLN: $f(\sqrt[3]{A}) = \left(\sqrt[3]{A}\right)^3 - A = A - A = 0$

(b) Show Newton's method applied to $f(x)$ leads to the iterative formula $x_{n+1} = \dfrac{2x_n + A/x_n^2}{3}$

SOLN: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 - A}{3x_n^2} = \dfrac{3x_n^3}{3x_n^2} - \dfrac{x_n^3 - A}{3x_n^2} = \dfrac{2x_n + A/x_n^2}{3}$

(c) Show that if $A = 8$ and $x_1 = 1$ then $x_3 = \dfrac{554}{225}$

SOLN: $x_2 = \dfrac{2 + 8}{3} = \dfrac{10}{3}$ so $x_3 = \dfrac{\frac{20}{3} + \frac{8}{100/9}}{3} = \dfrac{\frac{20}{3} + \frac{18}{25}}{3} \cdot \dfrac{75}{75} = \dfrac{500 + 54}{225} = \dfrac{554}{225}$