Math 1A **Name (Print):** ______________________

Chapter 3 Test-Solutions

Write all responses on separate paper. Show your work in detail for credit. No calculators.

1. (12 points) Compute the limit by interpreting it as the limit of a difference quotient,
$f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{a\to x} \dfrac{f(x)-f(a)}{x-a}$

   (a) $\lim_{x\to 2} \dfrac{x^5-x^4-16}{x-2}$

   SOLN: Let $f(x)=x^5-x^4$ then $f(2)=16$ and so $\lim_{x\to 2} \dfrac{x^5-x^4-16}{x-2} = \lim_{x\to 2}\dfrac{f(x)-f(2)}{x-2}$
   $= f'(2) = 5x^4-4x^3\Big|_{x=2} = 80-32 = 48$

   (b) $\lim_{x\to 0} \dfrac{\ln|\sec x|}{x}$

   SOLN: Let $f(x)=\ln|\sec x|$ then $f(0)=0$ and so $\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0}$
   $= f'(0) = \dfrac{du}{dx}\dfrac{d}{du}\ln u\Big|_{x=0} = \sec x\tan x\dfrac{1}{\sec x}\Big|_{x=0} = \tan(0) = 0$

2. (12 points) Use the **definition** of the derivative (that is, $f'(x) \equiv \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$) to compute $\dfrac{d}{dx}\cos(x)$

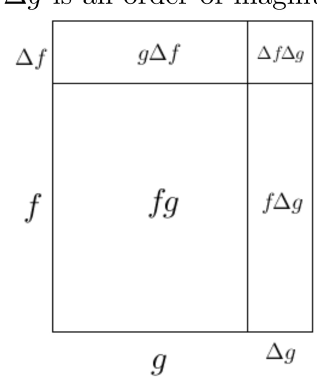
   SOLN: $\lim_{h\to 0}\dfrac{\cos(x+h)-\cos(x)}{h} = \lim_{h\to 0}\dfrac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}$
   $= \lim_{h\to 0}\dfrac{\cos(x)(\cos(h)-1)}{h} - \dfrac{\sin(x)\sin(h)}{h} = \cos(x)\lim_{h\to 0}\cancelto{0}{\dfrac{\cos(h)-1}{h}} - \sin(x)\lim_{h\to 0}\cancelto{1}{\dfrac{\sin h}{h}} = -\sin(x)$

3. (12 points) State each rule and give a compelling justification for it.

   (a) The product rule.

   SOLN: The product rule says that rate of change of a product of function comes in two terms, the value of the first function times the rate of change of second plus the value of the second function times the rate of change of first function. That is, $\dfrac{d}{dx}(f(x)g(x)) = f(x)g'(x)+f'(x)g(x)$. A good way to understand why is to think of the product as the area of a rectangle with dimensions $f(x)$ and $g(x)$. The infinitesimal growth of Δg on the edge of length f has infinitesimal area $f\Delta g$ and the infinitesimal growth of Δf on the edge of length g has infinitesimal area $g\Delta f$. The growth in the corner, $\Delta f\Delta g$ is an order of magnitude smaller.

   We write $\dfrac{d}{dx}(f(x)g(x)) = \dfrac{dA}{dx} = \lim_{\Delta x\to 0}\dfrac{f\Delta g + g\Delta f + \Delta f\Delta g}{\Delta x}$
   $= \lim_{\Delta x\to 0}\dfrac{f\Delta g}{\Delta x} + \lim_{\Delta x\to 0}\dfrac{g\Delta f}{\Delta x} + \lim_{\Delta x\to 0}\dfrac{\Delta f\Delta g}{\Delta x}$
   $= f(x)\lim_{\Delta x\to 0}\dfrac{\Delta g}{\Delta x} + g(x)\lim_{\Delta x\to 0}\dfrac{\Delta f}{\Delta x} + \lim_{\Delta x\to 0}\Delta f\lim_{\Delta x\to 0}\dfrac{\Delta g}{\Delta x}$
   $= f(x)g'(x)+g(x)f'(x)+0$



   (b) The chain rule.

   SOLN: The rate of change of a composite function of x is a product of the rate of change of the outer function per change in the inner function times the rate of change of the inner function per change in x: $\dfrac{d}{dx}f(g(x)) = \dfrac{df}{dg}\dfrac{dg}{dx}$. More informally $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ where $u=g(x)$. More formally, $\dfrac{dy}{dx} = \lim_{h\to 0}\dfrac{f(g(x+h))-f(g(x))}{h} = \lim_{h\to 0}\dfrac{f(g(x+h))-f(g(x))}{h} = \lim_{h\to 0}\dfrac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\dfrac{g(x+h)-g(x)}{h}$ Let $u=g(x)$ and $u+k=g(x+h)$.
   Then we can write this as $\lim_{k\to 0}\dfrac{f(u+k)-f(u)}{k}\lim_{h\to 0}\dfrac{g(x+h)-g(x)}{h}$

That's feels a bit like a trick, and not much different that multiplying by $1 = \frac{du}{du}$ and rearranging the factors. How about this? If the function is differentiable then both the outer function and the inner function are differentiable (is that really true?). Any differential function can be linearized as $f(x) = f(a) + f'(a)(x-a)$ and the composition of two linear function will multiply their slopes (multiply their rates of change.) That is, if $f(x) = ax+b$ and $g(x) = cx+d$ then $f(g(x)) = f(cx+d) = a(cx+d)+b = acx+ad+b$ so $\frac{d}{dx}f(g(x)) = ac$

4. (12 points) Suppose that $g(\pi) = 2$ and $g'(\pi) = -3$. Evaluate $\frac{d}{dx}\left(\frac{x}{g(x)}\right)\Big|_{x=\pi}$.

   SOLN: By the quotient rule, $\frac{d}{dx}\left(\frac{x}{g(x)}\right)\Big|_{x=\pi} = \frac{g(x)-xg'(x)}{(g(x))^2}\Big|_{x=\pi} = \frac{g(\pi)-\pi g'(\pi)}{(g(\pi))^2} = \boxed{\frac{2+3\pi}{4}}$

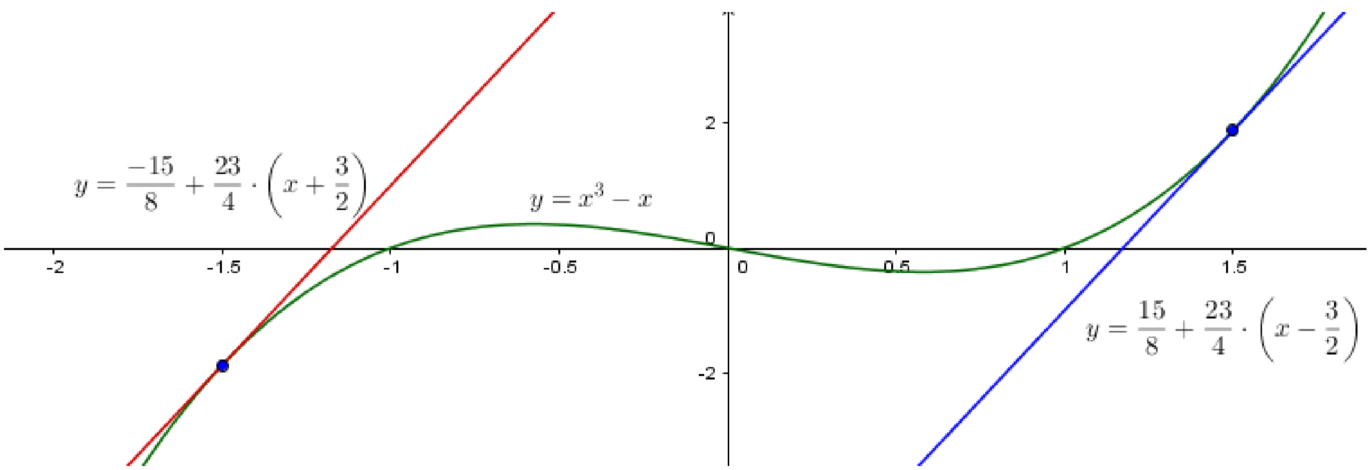
5. Find the equation(s) of the tangent line(s) for the curve $y = x^3 - x$ which are parallel to $23x - 4y = 17$

   SOLN: The slope of the line tangent to $y = x^3 - x$ at x is $y' = 3x^2 - 1$ which must match the slope of $23x - 4y = 17$ which is $m = \frac{23}{4}$. Solving $3x^2 - 1 = \frac{23}{4} \Leftrightarrow 3x^2 = \frac{23}{4} \Leftrightarrow x^2 = \frac{9}{4} \Leftrightarrow x = \pm\frac{3}{2}$ Thus the equations are

   $\boxed{y \pm \frac{15}{8} = \frac{23}{4}\left(x \pm \frac{3}{2}\right)}$

   In Sagemath, the following commands will produce a graph like the one shown:

```
f=x^3-x
P1=plot(f,(x,-2,2))
P1+=plot(15/8+23/4*(x-3/2),(x,0,2))
P1+=plot(-15/8+23/4*(x+3/2),(x,-2,0))
show(P1)
```



6. (12 points) A particle moves along a straight line with displacement $s(t)$, velocity $v(t)$, and acceleration $a(t)$. Show that

$$a(t) = v(t)\frac{dv}{ds}$$

   Explain the difference between the meanings of the derivatives dv/dt and dv/ds.

   SOLN: $a(t) = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{ds}{dt}\frac{dv}{ds} = v(t)\frac{dv}{ds}$

7. (16 points) Compute the derivative function for each.

   (a) $f(x) = \ln\left(\frac{1}{\cos(x)}\right)$

   SOLN: Let $u = \frac{1}{\cos(x)} = \sec(x)$ then $\frac{d}{dx}f(x) = \frac{du}{dx}\frac{d}{du}\ln(u) = \sec(x)\tan(x)\frac{1}{u} = \boxed{\tan(x)}$

   (b) $f(x) = \ln(1+\cos(x))$

   SOLN: Let $u = 1+\cos(x)$ then $\frac{d}{dx}f(x) = \frac{du}{dx}\frac{d}{du}\ln(u) = -\sin(x)\frac{1}{1+\cos(x)} = \boxed{-\frac{\sin(x)}{1+\cos(x)}}$

8. (10 points) Let

$$f(x) = \begin{cases} x-1 & : x \le 1 \\ a\cos(\frac{\pi}{2}x^2) + b & : 1 < x \end{cases}$$

Find values of a and b so that f is a differentiable function.
Hint: f must be both continuous and differentiable at $x = 1$.
SOLN: First, require that it be continuous, so that $\lim_{x\to 1} x - 1 = 0 = \lim_{x\to 1} a\cos(\frac{\pi}{2}x^2) + b = b$, so $\boxed{b = 0}$.
Now make sure it's smooth (differentiable): $\lim_{x\to 1^-} f'(x) = 1 = \lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+}(-a\pi x\sin(\frac{\pi}{2}x^2)) = -a\pi$ so $\boxed{a = -\frac{1}{\pi}}$

9. (12 points) A population of protozoa develops with a constant relative growth rate of 0.1 per member per hour. On day zero the population consists of two members. Find an expression for the population size after ten hours.
SOLN: $P(t) = 2e^{0.1t}$ gives the population after t hours. After 10 hours there will be $P(10) = 2e \approx 2 \cdot 2.72 \approx 5$ protozoa.

10. (14 points) The radius of a sphere is increasing at a rate of 2 meters per second. How fast is the surface area increasing when the radius is 10 meters? *Hint* The surface area of a sphere of radius r is $4\pi r^2$.
SOLN: We want the time rate of change in the surface area $= \frac{d}{dt}A = \frac{d}{dt}4\pi r^2 = 8\pi r\frac{dr}{dt}$. We know $\frac{dr}{dt} = 2\frac{\text{m}}{\text{sec}}$ so when $r = 10$m, $\frac{dA}{dt} = 8\pi(10)(2) = 160\pi\frac{\text{m}^2}{\text{sec}}$

11. Use a linear approximation (or differentials) to estimate the given number.
   (a) $\sin(3)$
   SOLN: We know $\sin(\pi) = 0$, so $\sin(3) \approx 0 + \cos(\pi)\Delta x \approx 0.1416$ This is a pretty good approximation to $\sin(3) \approx 0.14112000805986722210074480280811$
   (b) $\sqrt{4.1}$
   SOLN: We know $\sqrt{4} = 2$, so $\sqrt{4.1} \approx 2 + \frac{1}{2\sqrt{4}}\Delta x \approx 2 + \frac{1}{4}\cdot 0.1 = 2 + \frac{1}{40} = 2.025$. This is a pretty good approximation to $\sqrt{4.1} \approx 2.0248456731316586933246902289901$