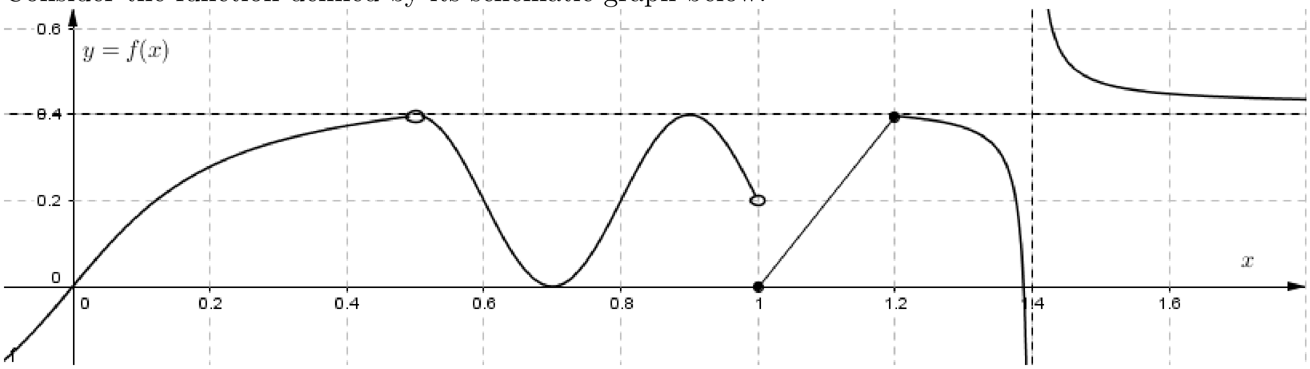
Math 1A **Name (Print):** ____________________

Final Exam

Write all responses on separate paper. Show your work in detail for credit. No calculators.

1. Consider the function defined by its schematic graph below.



   (a) Find each limit or right "DNE" if the limit does not exist.

   i $\lim_{x\to 0.5} f(x) = 0.4$ ii $\lim_{x\to 1^-} f(x) = 0.2$ iii $\lim_{x\to 1} f(x)$ DNE

   (b) Find all the discontinuities and classify each as either a removable discontinuity, a jump discontinuity or a vertical asymptote. F has a removable discontinuity where $x = 0.5$, a jump discontinuity where $x = 1$ and a vertical asymptote along $x = 1.4$

   (c) Find $f'(1.1)$ ANS: The derivative gives the instantaneous rate of change, or the slope of the tangent line. Since the function is linear here $f'(1.1) =$ slope of the line $= \dfrac{0.4-0}{1.2-1} = 2$

   (d) Solve for x: $f'(x) = 0$. ANS: There are two points where the tangent line is horizontal: $(0.7, 0)$ and $(0.9, 0.4)$.

2. Use the **definition** of the derivative (that is, $f'(x) \equiv \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$) to show that $\dfrac{d}{dx}x^3 = 3x^2$.

   SOLN: $\lim_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} 3x^2 + \cancel{3xh + h^2}^{\,0}$

3. Suppose that $g(2) = 3, g(3) = 4, g'(3) = 1$, and $g'(2) = -1$. Evaluate $\left.\dfrac{d}{dx} g(g(x))\right|_{x=2}$.

   SOLN: Let $u = g(x)$. Then $\left.\dfrac{d}{dx} g(g(x))\right|_{x=2} = \left.\dfrac{du}{dx}\dfrac{d}{du} g(u)\right|_{x=2} = g'(2)g'(g(2)) = -1 \cdot g'(3) = -1 \cdot 1 = -1$

4. Let $f(x) = x^3 - x^2$ on $[0, 2]$.

   (a) Explain why the function satisfies the conditions of the Mean Value Theorem.
   SOLN: The function, f is polynomial and thus differentiable and continuous on all intervals.

   (b) Find all values of c which satisfy the conclusion of the Mean Value Theorem.
   SOLN: The slope of the line tangent to $y = x^3 - x^2$ at x is $y' = 3x^2 - 2x$. The slope of the secant line from $(0, f(0))$ to $(2, f(2)$ is $m = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{4-0}{2-0} = 2$. Solving $f'(c) = 2 \Leftrightarrow 3c^2 - 2c = 2 \Leftrightarrow c^2 + \frac{2}{3}x = \frac{2}{3} \Leftrightarrow$ $\left(c + \frac{1}{3}\right)^2 = \frac{7}{9} \Leftrightarrow c = -\dfrac{1}{3} \pm \dfrac{\sqrt{7}}{3}$. Only $\boxed{c = \dfrac{-1+\sqrt{7}}{3} \in [0, 2]}$

5. Let $f(x) = \sin^3(2x)$ so that $f\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8} \approx 0.6495$

   (a) Find linear function for the tangent line to $y = f(x)$ at $x = \frac{\pi}{6}$.
   SOLN: The slope of the line is $f'\left(\frac{\pi}{6}\right) = 3\sin^2(2x)\cos(2x)\cdot 2\Big|_{x=\frac{\pi}{6}} = 6\left(\sin\left(\frac{\pi}{3}\right)\right)^2\cos\left(\frac{\pi}{3}\right) = \frac{9}{4}$. Plugging into the point-slope formula we have $L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \boxed{\frac{3\sqrt{3}}{8} + \frac{9}{4}\left(x - \frac{\pi}{6}\right) = \frac{9}{4}x - \frac{3\pi - 3\sqrt{3}}{8}}$

   (b) Use the tangent line to approximate $f\left(\frac{1}{2}\right)$. Note that $\frac{\pi}{6} \approx 0.5235$
   SOLN: $f\left(\frac{1}{2}\right) \approx L\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{8} + \frac{9}{4}\left(\frac{1}{2} - \frac{\pi}{6}\right) \approx 0.6495 - 2.25(0.0235) \approx 0.6498 - 0.0529 = \boxed{0.5967}$

6. Let
$$f(x) = \begin{cases} \sin(\pi x) & : x \le \frac{1}{2} \\ (x-h)^2 + k & : x > \frac{1}{2} \end{cases}$$

   (a) Find values of h and k so that f is a differentiable function, and demonstrate that it is differentiable.
   We require continuity, so $\sin\left(\frac{\pi}{2}\right) = \left(\frac{1}{2} - h\right)^2 + k \Rightarrow \left(\frac{1}{2} - h\right)^2 + k = 1$.
   We also require that the derivative from the left agrees with the derivative from the right:
   $\pi\cos\left(\frac{\pi}{2}\right) = 2\left(\frac{1}{2} - h\right) \Rightarrow \boxed{h = \frac{1}{2}}$. From the continuity requirement then, $\boxed{k = 1}$

   (b) Do the second derivatives agree at $x = \frac{1}{2}$?
   SOLN: No, from the left, $f''(x) \to -\pi^2$ while from the right, $f''(x) \to 2$.

7. The circumference of a circle is increasing at a rate of 2 meters per second. How fast is the radius increasing when the radius is 4960 kilometers? *Hint* Circumference $= 2\pi r$.
   SOLN: $\frac{d}{dt}C = \frac{d}{dt}(2\pi r) = 2 \Leftrightarrow 2\pi\frac{dr}{dt} = 2 \Leftrightarrow \frac{dr}{dt} = \frac{1}{\pi}$ meters per second, regardless of what r is.

8. (14 points) Use the methods of constrained optimization to find the dimensions of the rectangle of largest area that can be inscribed in the unit circle. That is, maximize the length*width subject to the rectangle fitting in a circle of radius 1.
   SOLN: Let (x, y) be a corner of the rectangle in the first quadrant and assume that the rectangle is oriented with its side parallel to the coordinate axes and centered at $(0, 0)$. Then the area of the rectangle is $A = (2x)(2y) = 4xy$. If the rectangle is inscribed in the unit circle, $x^2 + y^2 = 1$, then $y = \sqrt{1 - x^2}$ and substituting, we have a the area as a function of a single variable: $A(x) = 4x\sqrt{1 - x^2} \Rightarrow A'(x) = 4\sqrt{1 - x^2} - \frac{4x^2}{\sqrt{1 - x^2}}$. The maximum occurs where $A'(x) = 0$, so $4\sqrt{1 - x^2} = \frac{4x^2}{\sqrt{1 - x^2}} \Leftrightarrow x^2 = 1 - x^2 \Leftrightarrow x = \frac{\sqrt{2}}{2}$. To be sure this is a maximum note that A is increasing for $x \in \left(0, \frac{\sqrt{2}}{2}\right)$ and decreasing for $x \in \left(\frac{\sqrt{2}}{2}, 1\right)$. Also, $A''(x) = \frac{-4x}{\sqrt{1 - x^2}} - \frac{8x\sqrt{1 - x^2} + \frac{8x^3}{\sqrt{1 - x^2}}}{1 - x^2} = \frac{-4x}{\sqrt{1 - x^2}} - \frac{8x}{(1 - x^2)^{3/2}} < 0$ for $x \in (0, 1)$ The maximum area is $A\left(\frac{\sqrt{2}}{2}\right) = 2$ square units.

9. (18 points) Compute

   (a) $\lim_{x \to 0^+} x \ln x$
   SOLN: $\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$ is an $\frac{\infty}{\infty}$ situation, $= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0$.

   (b) $\lim_{x \to \infty} \frac{e^x}{x^{100}} = \lim_{x \to \infty} \frac{e^x}{100x^{99}} = \lim_{x \to \infty} \frac{e^x}{100 \cdot 99x^{98}} = \cdots = \lim_{x \to \infty} \frac{e^x}{100!} = \infty$

10. Use the guidelines to sketch the curve $y = xe^{-x^2/2}$

(a) What is the domain of f ? ANS: All real numbers, domain $= \mathbb{R}$

(b) What are the intercepts of f ? ANS: The only intercept is $(0, 0)$

(c) What symmetry does f have? ANS: Odd (origin) symmetry.

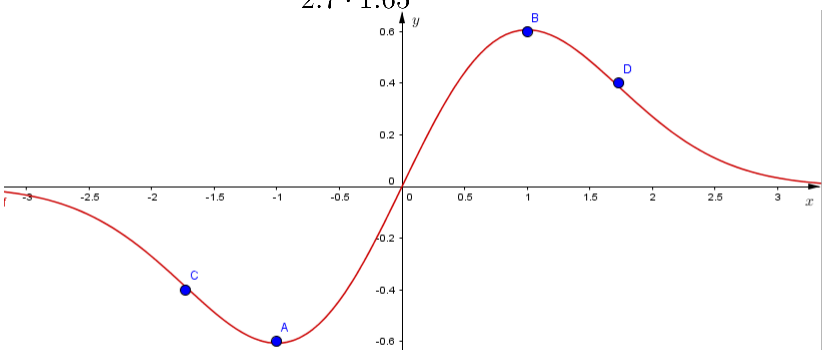
(d) What asymptote(s) ? $y = 0$ is a horizontal asymptote.

(e) On what interval(s) is f increasing/decreasing?
ANS: $y' = e^{-x^2/2} - x^2e^{-x^2/2} = (1 - x^2)e^{-x^2/2} > 0 \Leftrightarrow x \in (-1, 1)$. So y is decreasing on $(-\infty, -1) \cup (1, \infty)$ and increasing on $(-1, 1)$.

(f) What are the extreme values of f ?

(g) On what intervals is f concave up/down ? Where are the inflection points?
$y'' = -2xe^{-x^2/2}+(1-x^2)xe^{-x^2/2} = x(x^2-3)e^{-x^2/2}$ changes sign at $(-\sqrt{3}, -\sqrt{3}e^{-3/2})$, $(0, 0)$ and $(\sqrt{3}, \sqrt{3}e^{-3/2})$

(h) Sketch a graph for f showing these features.
Without a calculator its difficult to get a very accurate approximation for y when $x = \pm 1$, but a rough estimate will do for sketching a graph. $y = \pm\dfrac{1}{\sqrt{e}} \approx \pm\dfrac{1}{\sqrt{2.7}} \approx \pm\dfrac{1}{1.65} \approx \pm 0.6$. When $x = \sqrt{3} \approx 1.7$, $y \approx \pm 1.7 \cdot e^{-1.5} \approx 1.7 \cdot \dfrac{1}{2.7 \cdot 1.65} \approx 0.4$, let's say. The graph then looks like



11. (14 points) Consider the integral function $F(x) = \int_1^x 1 - t^2 dt$

(a) Evaluate $F(2)$ using the Fundamental Theorem of Calculus.
SOLN: $F(2) = \int_1^2 1 - t^2 dt = t - \frac{1}{3}x^3\Big|_1^2 = 2 - \frac{8}{3} - \left(1 - \frac{1}{3}\right) = \dfrac{-4}{3}$

(b) Find $F'(x)$ using the Fundamental Theorem of Calculus.
SOLN: $F'(x) = 1 - x^2$

(c) Approximate $F(2)$ by a Riemann sum of $n = 2$ rectangles of equal width and with midpoints as sample points.
SOLN: $\Delta x = \dfrac{2-1}{2} = \dfrac{1}{2}$ and $x_1^* = \frac{5}{4}, x_2^* = \frac{7}{4}$ so that $f(x_1^*) = f\left(\frac{5}{4}\right) = 1 - \frac{25}{16} = -\frac{9}{16}$ and $f(x_2^*) = f\left(\frac{7}{4}\right) = 1 - \frac{49}{16} = -\frac{33}{16}$ whence $M_2 = \dfrac{1}{2}\left(-\dfrac{7}{16} - \dfrac{33}{16}\right) = -\dfrac{1}{2} \cdot \dfrac{5}{2} = -\dfrac{5}{4}$

(d) Find $F(2)$ using the definition of the Riemann integral. That is, as the $\lim_{n\to\infty} R_n$ where R_n is a Riemann sum over a partition with n subintervals of equal length and with right endpoints as sample points. Find Δx, x_i, $f(x_i)$, plug in and simplify the limit.
SOLN: $\Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}$ and $x_i = 1 + \frac{i}{n}$ so that $F(2) = \lim_{n\to\infty} \sum_{i=1}^n f(1 + \frac{i}{n})\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n 1 - (1 + \frac{i}{n})^2 = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n -\frac{2i}{n} - \frac{i^2}{n^2} = -\lim_{n\to\infty}\left(\frac{2}{n^2}\sum_{i=1}^n i + \frac{1}{n^3}\sum_{i=1}^n i^2\right) = -\lim_{n\to\infty}\left(\frac{2}{n^2}\frac{n^2+n}{2} + \frac{1}{n^3}\frac{2n^3+o(n^2)}{6}\right) = 1 + \frac{1}{3} = \boxed{\dfrac{4}{3}}$