Math 1A – Chapter 4 Test – Fall '10 Name______________________

Show your work for credit. Write all responses on separate paper. No calculators.

1. Find each of the following limits:

   a. $\lim_{x\to 0}\frac{\tan 3x}{\sin 2x}$

   b. $\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}$

   c. $\lim_{x\to 0}\frac{e^{5x}-1-\sin x}{x^2+x}$

2. Consider $y=x^3e^{2x}$.

   a. Find all local extrema for y.

   b. Find all the inflection points for y.

   c. Sketch a graph for y showing these features.

3. For what values of a and b does the function $f(x)=\frac{ax}{b+x^2}$ have a maximum at $f(1)=2$?

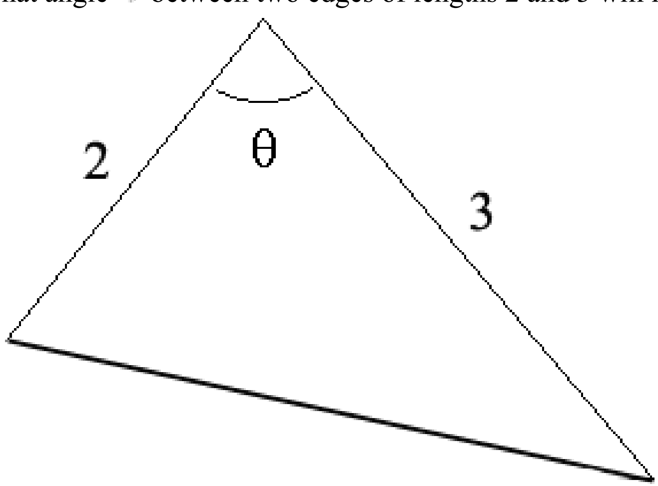
4. If $y = x^2 - x + 1$, what value will minimize the product xy on the interval [0, 2] ?

5. Find the point on the parabola, $y = x^2$, that is closest to the point (3,0).

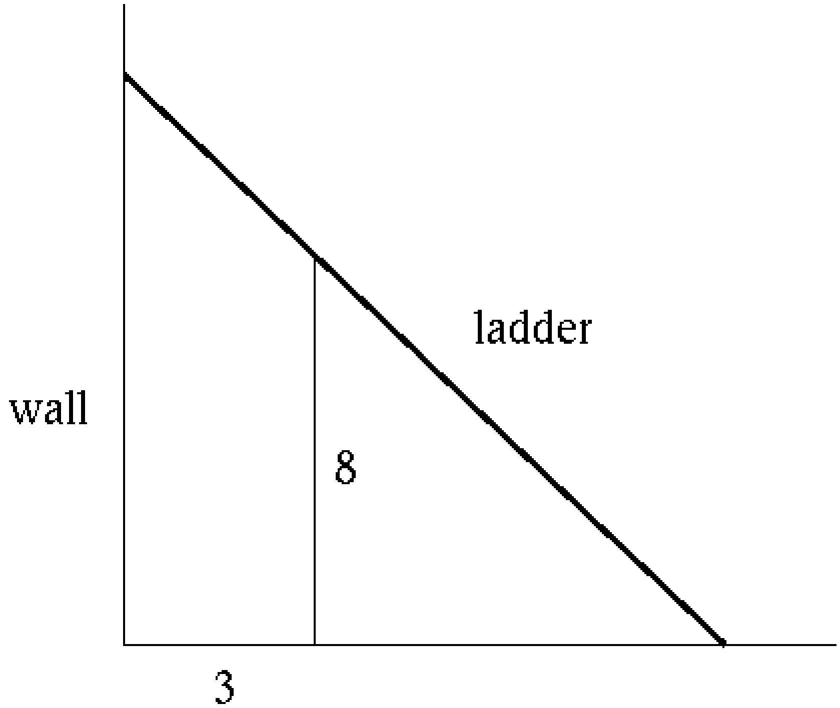
6. Consider all triangles whose sides are formed by a line passing through the point (8/3, 1) and both the x- and y-axes. Find the dimensions of the triangle with the shortest hypotenuse.

7. A triangle has vertices at (0,0), (a,0) and (b,c). What are the coordinates of the point, P, such that the sum of the squares of the distances from P to the vertices of the triangle is minimized?

8. What angle θ between two edges of lengths 2 and 3 will result in a triangle with the largest area ?



9. Find the length of the shortest ladder that will reach over an 8-ft. high fence to a large wall which is 3 ft. behind the fence.



Math 1A – Chapter 4 Test (part 2) Solutions – Fall '10

1. Find each of the following limits:

   a. $\lim_{x\to 0}\frac{\tan 3x}{\sin 2x}=\lim_{x\to 0}\frac{3\sec^2 3x}{2\cos 2x}=\frac{3}{2}$

   b. $\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}=\lim_{x\to 0}\frac{-m\sin mx-n\sin nx}{2x}=\lim_{x\to 0}\frac{-m^2\cos mx-n^2\cos nx}{2}=-\frac{m^2+n^2}{2}$

   c. $\lim_{x\to 0}\frac{e^{5x}-1-\sin x}{x^2+x}=\lim_{x\to 0}\frac{5e^{5x}-\cos x}{2x+1}=4$

2. Consider $y=x^3e^{2x}$.

   a. Find all local extrema for y.

   SOLN: $y'=2x^3e^{2x}+3x^2e^{2x}=x^2(2x+3)e^{2x}=0 \Leftrightarrow x=0$ or $x=-\frac{3}{2}$ Since $y'<0$ for $x<\frac{-3}{2}$ and y' ≥ 0 for $x>\frac{-3}{2}$ (note that y' doesn't change sign at $x = 0$).

   Thus $y=\left(\frac{-3}{2}\right)^3 e^{2(-3/2)}=\frac{-27}{8e^3}\approx -0.168$ is the absolution minimum. There are no other extrema.

   b. Find all the inflection points for y.

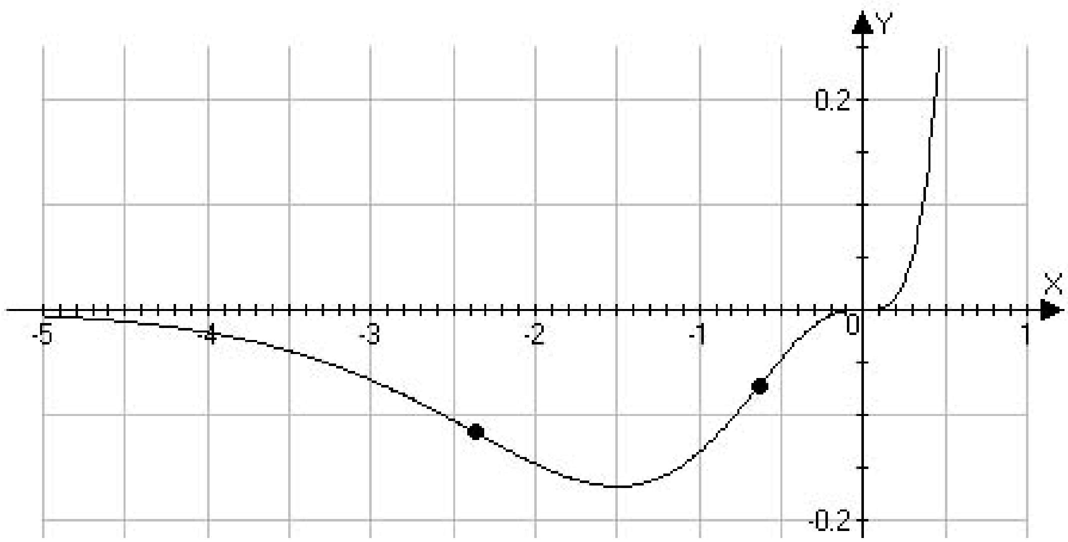
   $$y'=\left(2x^3+3x^2\right)e^{2x}\Rightarrow$$

   SOLN: $y''=2\left(2x^3+3x^2\right)e^{2x}+\left(6x^2+6x\right)e^{2x}=2x\left(2x^2+6x+3\right)e^{2x}$

   $$=4x\left[\left(x+\frac{3}{2}\right)^2-\frac{3}{4}\right]e^{2x}$$

   Thus the second derivative changes sign where $x = 0$ and where $\left(x+\frac{3}{2}\right)^2=\frac{3}{4}\Leftrightarrow \boxed{x=-\frac{3}{2}\pm\frac{\sqrt{3}}{2}}$

   c. Sketch a graph for y showing these features.



3. For what values of *a* and *b* does the function $f(x)=\frac{ax}{b+x^2}$ have a maximum at $f(1)=2$?

SOLN: Impose the two conditions $f(1)=2$ and $f'(1)=0$ to get a system of equations you can solve for *a* and *b*. That is, $f(1)=\frac{a}{b+1}=2 \Leftrightarrow a=2(b+1)$ and

$$f'(x)=\frac{(b+x^2)(ax)' - ax(b+x^2)'}{(b+x^2)^2}=\frac{a(b+x^2)-2ax^2}{(b+x^2)^2}=\frac{a(b-x^2)}{(b+x^2)^2}$$ so that $f'(1)=0 \Rightarrow a(b-1)=0$.

Now since $a \neq 0$ we must have $b = 1$ whence $a = 4$.

4. If $y = x^2 - x + 1$, what value will minimize the product xy on the interval [0, 2] ?
SOLN: Substituting for y, $xy = x(x^2 - x + 1) = x^3 - x^2 + x = P(x)$. To minimize the product we look for critical points, that is where $P'(x) = 3x^2 - 2x + 1 = 3(x - 1/3)2 + 2/3 \geq 2/3$ for all x. This means the function is increasing everywhere so the minimum product must be at the endpoint $P(0) = 0$.

5. Find the point on the parabola, $y = x^2$, that is closest to the point (3,0).
SOLN: There are at least two good approaches to the problem: you can use the distance formula or the fact that the line tangent to the curve at the closest point will be perpendicular to the line segment connecting it to (3,0). Using the distance formula we have $D^2 = (3 - x)^2 + (0 - x^2)^2 = x^4 + x^2 - 6x + 9$.
Thus $2DD' = 4x^3 + 2x - 6 = 2(x - 1)(2x^2 + 2x + 3) = 0$ if $x = 1$. So point on $y = x^2$ that's closest to (3,0) is (1,1).

6. Consider all triangles whose sides are formed by a line passing through the point (8/3, 1) and both the x- and y-axes. Find the dimensions of the triangle with the shortest hypotenuse.
SOLN: The first question is what to use as a control parameter? There are several good answers to that question, including the slope of the line and the angle that slope makes with the *x* axis.
Let m = the slope of the line. Then the equation for the line is $y - 1 = m(x - 8/3)$ and the intercepts are then $(0, 1 - 8m/3)$ and $(8/3 - 1/m)$ mean that the square of the length of the hypotenuse is

$$D^2=\left(1-\frac{8m}{3}\right)^2+\left(\frac{8}{3}-\frac{1}{m}\right)^2=\left(1-\frac{8m}{3}\right)^2+\frac{1}{m^2}\left(\frac{8m}{3}-1\right)^2=\left(1-\frac{8m}{3}\right)^2\left(1+\frac{1}{m^2}\right).$$ Thus

$$2DD'=-\frac{16}{3}\left(1-\frac{8m}{3}\right)\left(1+\frac{1}{m^2}\right)+\left(1-\frac{8m}{3}\right)^2\left(\frac{-2}{m^3}\right)=\left(1-\frac{8m}{3}\right)\left[-\frac{16}{3}\left(1+\frac{1}{m^2}\right)-\frac{2}{m^3}\left(1-\frac{8m}{3}\right)\right]=0$$ So

either $1-\frac{8m}{3}=0 \Leftrightarrow m=\frac{3}{8}$ or $-\frac{16}{3}\left(1+\frac{1}{m^2}\right)-\frac{2}{m^3}\left(1-\frac{8m}{3}\right)=0 \Leftrightarrow 8m^3+3=0 \Leftrightarrow m=\frac{-\sqrt[3]{3}}{2}$

$$D^2=\left(1+\frac{4\sqrt[3]{3}}{3}\right)^2\left(1+\frac{4}{\sqrt[3]{9}}\right)=\left(1+\frac{4\sqrt[3]{3}}{3}\right)^2\left(1+\frac{4\sqrt[3]{3}}{3}\right)=\left(1+\frac{4\sqrt[3]{3}}{3}\right)^3$$ So the dimensions of the triangle are

hypotenuse = $\left(1+\frac{4\sqrt[3]{3}}{3}\right)^{3/2}$, vertical leg $=\frac{3+4\sqrt[3]{3}}{3}$ and horizontal leg = $\frac{8}{3}+\frac{2}{\sqrt[3]{3}}=\frac{8+2\sqrt[3]{9}}{3}$

Alternatively, Let θ = the acute angle the hypotenuse forms with the x-axis. Then we can break up the hypotenuse = $a + b$ where $\sin\theta=\frac{1}{a}$ and $\cos\theta=\frac{8}{3b}$. So hypotenuse = $a+b=f(\theta)=\frac{1}{\sin\theta}+\frac{8}{3\cos\theta}$ and the extrema are where $f'(\theta)=0 \Leftrightarrow f'(\theta)=\frac{\cos\theta}{\sin^2\theta}-\frac{8\sin\theta}{3\cos^2\theta}=\frac{3\cos^3\theta-8\sin^3\theta}{3\sin^2\theta\cos^2\theta}=0$

So that $\tan^3\theta=\frac{3}{8} \Leftrightarrow \tan\theta=\frac{\sqrt[3]{3}}{2}$. This is the same result we got through somewhat greater effort above.

7. A triangle has vertices at (0,0), (a,0) and (b,c). What are the coordinates of the point, P, such that the sum of the squares of the distances from P to the vertices of the triangle is minimized?
SOLN: The sum of the squares of distances is
$S = (x-0)^2 + (y-0)^2 + (x-a)^2 + (y-0)^2 + (x-b)^2 + (y-c)^2 = 3x^2 - 2(a+b)x + 3y^2 - 2cy + a^2 + b^2 + c^2$.
There are amny approaches to minimizing this distance. One is to set both $\frac{dy}{dx}=0$ and $\frac{dx}{dy}=0$.
The other is to complete the squares for both x and y. Completing the squares is relatively straight forward:
$S = 3x^2 - 2(a+b)x + 3y^2 - 2cy + a^2 + b^2 + c^2$.

$$\begin{aligned}
S &= 3x^2-2(a+b)x+3y^2-2cy+a^2+b^2+c^2 \\
&= 3\left[x^2-\frac{2(a+b)}{3}x+\left(\frac{a+b}{3}\right)^2\right]+3\left[y^2-\frac{2}{3}cy+\left(\frac{c}{3}\right)^2\right]+a^2+b^2+c^2-3\left(\frac{a+b}{3}\right)^2-3\left(\frac{c}{3}\right)^2 \\
&= 3\left(x-\frac{a+b}{3}\right)^2+3\left(y-\frac{c}{3}\right)^2+a^2+b^2+c^2-\frac{a^2+2ab+b^2}{3}-\frac{c^2}{3} \\
&= 3\left(x-\frac{a+b}{3}\right)^2+3\left(y-\frac{c}{3}\right)^2+\frac{2}{3}\left(a^2-ab+b^2+c^2\right)
\end{aligned}$$

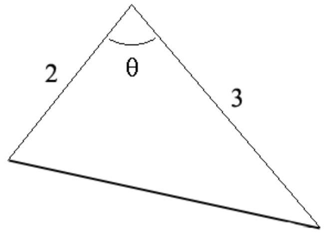
is clearly minimized when $x=\frac{a+b}{3}$ and $y=\frac{c}{3}$

Alternatively, one can differentiate implicitly with respect to x and solve $y' = 0$. Then do the same for $x' = 0$.

$$\frac{d}{dx}S=6x-2(a+b)+6y\frac{dy}{dx}-2c\frac{dy}{dx}=0 \Leftrightarrow \frac{dy}{dx}=\frac{2(a+b)-6x}{6y-2c}.$$

So $y' = 0$ when $x=\frac{a+b}{3}$ and $x' = 0$ when $y=\frac{c}{3}$. That's some pretty neat calculus, huh?

8. What angle θ between two edges of lengths 2 and 3 will result in a triangle with the largest area ?



SOLN: The area of the triangle is ½ Base * Altitude. Taking 2 as the base the altitude is 3sinθ so the area is $A(\theta)=\frac{1}{2}(2)(3\sin\theta)=3\sin\theta$ which has an obvious maximum of 3. Go ahead set the derivative to zero and solve for θ to verify this.

9. Find the length of the shortest ladder that will reach over an 8-ft. high fence to a large wall which is 3 ft. behind the fence.