Math 1A – Chapter §4.1-4.3 Test – Fall '10 Name________________________

Show your work for credit. Write all responses on separate paper. No calculators.

1. Genie wants to find the absolute max and min of $f(x) = 2x^3 - 7x^2 - 12x + 2$ on the interval [–1,4]
   a. What theorem would she apply? Why would this theorem work for $f(x)$?
   b. Show how to use the Extreme Value Theorem and the closed interval method to find the local and absolute max and min for $f(x)$ on the interval [–1,4].

2. Let $f(x) = \begin{cases} \dfrac{x^2}{\sin x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$
   a. Show that *f*(*x*) satisfies the conditions of the extreme value theorem on [-3,3]. *Hint*: You can use L'Hospital's rule, if it makes it easier

   b. Use the "closed interval method" to find all max and min on the interval [0,*e*] and classify each as either local or absolute.

3. Consider the function $f(t) = e^{2\cos t}$ on the interval $t \in [0, 2\pi]$.
   a. Find a formula for the first derivative and use it to find where is *f* increasing on this interval. Write your answer using interval notation.

   b. Where is *f* concave up on this interval? *Hint*: Use the Pythagorean identity to get an equation quadratic in sin(*x*).

4. Consider $f(x) = (x-3)^3 (x-1)^4$
   a. Compute $f'(x)$ and factor the result to find where *f* is increasing and where it's decreasing.
   b. Find the local and absolute max and min of *f* on the interval [0,4].

5. Consider $f(x) = \ln x$ on the interval [1,e]. Find *c* in (1,e) so that $f'(c)(e-1) = 1$.

6. Consider $f(x) = \sqrt[3]{x}$ on [–1,1].
   a. Explain why *f* doesn't satisfy the conditions of the mean value theorem on that interval.
   b. Nevertheless, find all values of *x* which satisfy the conclusion of the mean value theorem on that interval and make a sketch to show what's going on. Hint: the graph of *f* is an "S" shape, in a way.

Math 1A – Chapter §4.1-4.3 Test Solutions – Fall '10

1. Genie wants to find the absolute max and min of $f(x)=2x^3-7x^2-12x+2$ on the interval [–1,4]

   a. What theorem would she apply? Why would this theorem work for $f(x)$?

   SOLN: Since $f(x)$ is continuous on [–1,4], the Extreme Value Theorem guarantees the existence of an absolute max and an absolute min.

   b. Show how to use the Extreme Value Theorem and the closed interval method to find the local and absolute max and min for $f(x)$ on the interval [–1,4].

   SOLN: Genie could use the "closed interval method." She'd start by finding the critical numbers where $f'(x)=6x^2-14x-12=0 \Leftrightarrow 3x^2-7x-6=(3x+2)(x-3)=0 \Leftrightarrow \boxed{x=-\frac{2}{3} \text{ or } 3}$ Both of these critical numbers are on the interior of the interval. Then she'd evaluate the function at these points. Writing $f(x)=((2x-7)x-12)x+2$ makes it a bit simpler to evaluate the function:

   $$f\left(-\frac{2}{3}\right)=\left(\left(-\frac{4}{3}-\frac{21}{3}\right)\left(-\frac{2}{3}\right)-12\right)\left(-\frac{2}{3}\right)+2=\left(\frac{50}{9}-\frac{108}{9}\right)\left(-\frac{2}{3}\right)+2=\frac{116}{27}+2=\frac{170}{27}=6.\overline{296}$$

   $$f(3)=((-1)(3)-12)(3)+2=(-15)(3)+2=-43$$

   Comparing these values with the endpoint values, $f(-1)=((-9)(-1)-12)(-1)+2=(-3)(-1)+2=5$ and $f(4)=((1)(4)-12)(4)+2=(-8)(4)+2=-30$ she'd conclude that the absolute min is at (3,–43) and the absolute max is $\left(-\frac{2}{3},\frac{170}{27}\right)$

2. Let $f(x)=\begin{cases}\dfrac{x^2}{\sin x} & \text{if} \quad x\neq 0\\ 0 & \text{if} \quad x=0\end{cases}$

   a. Show that $f(x)$ satisfies the conditions of the extreme value theorem on [0,e].

   SOLN: Both x^2 and sin x are continuous on [0,e], though there is a problem at 0, the only place in the interval where sin x = 0. However, $\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{x^2}{\sin x}=\left(\lim_{x\to 0}x\right)\left(\lim_{x\to 0}\frac{x}{\sin x}\right)=0(1)=0=f(0)$, so f is continuous on the interval. Now $f'(x)=\dfrac{2x\sin x-x^2\cos x}{\sin^2 x}$ exists on (0,e) so f meets the conditions of the Mean Value Theorem.

   b. Use the "closed interval method" to find all max and min on the interval [0,e] and classify each as either local or absolute.

   SOLN: $f'(x)=\dfrac{2x\sin x-x^2\cos x}{\sin^2 x}=0 \Leftrightarrow 2\sin x-x\cos x=0 \Leftrightarrow 2\tan x=x$ so there are no critical numbers in [0,e] since 2tan $x > x$ on $\left(0,\frac{\pi}{2}\right)$ and tan x is negative on $\left(\frac{\pi}{2},e\right)$. In fact,

   $f'(x)=\dfrac{2x\sin x-x^2\cos x}{\sin^2 x}>0$ on (0,e) so the minimum occurs at (0,0) and the maximum at $\left(0,\dfrac{e^2}{\sin e}\right)$

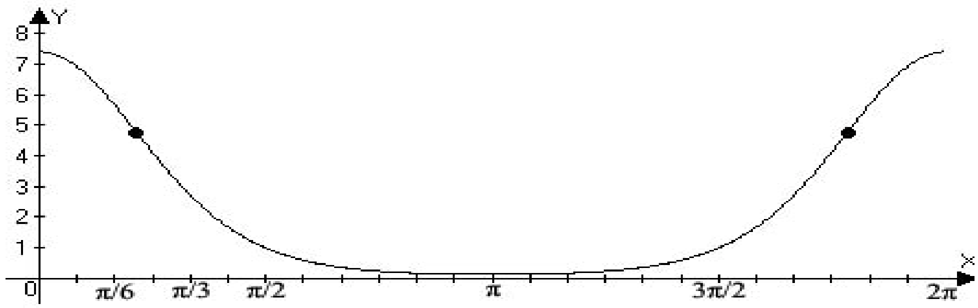
3. Consider the function $f(t)=e^{2\cos t}$ on the interval $0<t<2\pi$.

   a. Find a formula for the first derivative and use it to find where is *f* increasing on this interval. Write your answer using interval notation.
   SOLN: $f'(t)=-2\sin t\, e^{2\cos t}>0 \Leftrightarrow -2\sin t>0 \Leftrightarrow \sin t<0$ shows that *f* is increasing on $(\pi,2\pi)$.

   b. Where is *f* concave up on this interval? *Hint*: Use the Pythagorean identity to get an equation quadratic in sin(*x*).

   a. $f''(t)=\left(4\sin^2 t-2\cos t\right)e^{2\cos t}>0 \Leftrightarrow 4\sin^2 t-2\cos t>0 \Leftrightarrow 4\left(1-\cos^2 t\right)-2\cos t>0$
   $\Leftrightarrow 2\cos^2 t+\cos t-2>0$
   $\Leftrightarrow 2\cos^2 t+\cos t-2>0 \Leftrightarrow t\in\left(\cos^{-1}\frac{\sqrt{17}-1}{4},2\pi-\cos^{-1}\frac{\sqrt{17}-1}{4}\right)\approx(0.675,5.61)$

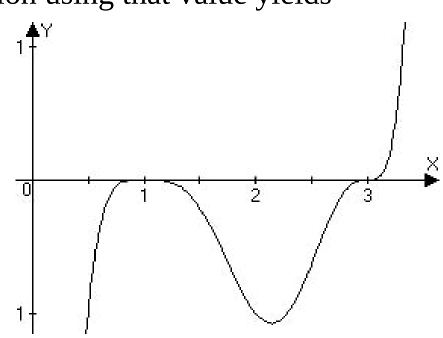
   Here's what this function looks like. The inflection points are plotted:



4. Consider $f(x)=(x-3)^3(x-1)^4$

   a. Compute $f'(x)$ and factor the result to find where *f* is increasing and where it's decreasing.
   SOLN:
   $f'(x)=3(x-3)^2(x-1)^4+4(x-3)^3(x-1)^3=(x-3)^2(x-1)^3\left(3(x-1)+4(x-3)\right)=(x-3)^2(x-1)^3(7x-15)$
   So $f'(x)<0$ only on $\left[1,\frac{15}{7}\right]$, meaning that *f* is decreasing only on $\left[1,\frac{15}{7}\right]$ and more or less increasing everywhere else.

   b. Find the local and absolute max and min of *f* on the interval [0,4].
   SOLN: The function is polynomial so it is differentiable everywhere, so the EVT applies and we can use the closed interval method. The critical numbers are *x* = 1 and $x=\frac{15}{7}$. Computing the function values at the endpoints and critical numbers yields $f(0)=(-3)^3(-1)^4=-27$, $f(1)=(1-3)^3(1-1)^4=0$,
   $f\left(\frac{15}{7}\right)=\left(\frac{15}{7}-3\right)^3\left(\frac{15}{7}-1\right)^4=\left(\frac{-6}{7}\right)^3\left(\frac{8}{7}\right)^4=\frac{8^4(-6)^3}{7^7}=\frac{-2^{15}3^3}{7^7}=\frac{-32768*27}{343*2401}=\frac{-884736}{823543}\approx-1.1$, note that this is slightly below $f(2)=-1$ and a first order linear approximation using that value yields
   $f\left(\frac{15}{7}\right)\approx f(2)+\frac{1}{7}f'(2)=-1-\frac{1}{7}\approx-1.14$ and since *f* is concave up in the neighborhood, this is an underestimate.
   $f(x)=(4-3)^3(4-1)^4=81$. Thus (0,–27) is an absolute min, (1,0) is a local max, $\left(\frac{15}{7},\frac{-884736}{823543}\right)$ is a local min and (4,81) is a global max. The graph at right corroborates this.



5. Consider $f(x) = \ln x$ on the interval [1,*e*]. Find *c* in (1,*e*) so that $f'(c)(e-1) = 1$.

SOLN: Note that the existence of such a *c* is guaranteed by the mean value theorem: *f* is differentiable on [1,*e*]. $f'(c) = \frac{f(e) - f(1)}{e-1} = \frac{1}{e-1} = \frac{1}{c} \Leftrightarrow \boxed{c = e-1}$

6. Consider $f(x) = \sqrt[3]{x}$ on [–1,1].

a. Explain why *f* doesn't satisfy the conditions of the mean value theorem on that interval.
SOLN: The function is not differentiable at $x = 0$.
There is a vertical tangent line at $x=0$

b. Nevertheless, find all values of *x* which satisfy the conclusion of the mean value theorem on that interval and make a sketch to show what's going on.
SOLN: Sketch is at right. The points where the slopes of the tangent lines are parallel to the secant line are found by solving $f'(x) = \frac{1}{3\sqrt[3]{x^2}} = \frac{\sqrt[3]{1} - \sqrt[3]{-1}}{1-(-1)} = \frac{2}{2} = 1$

$\Leftrightarrow \sqrt[3]{x^2} = \frac{1}{3} \Leftrightarrow x = \pm\frac{\sqrt{3}}{9}$