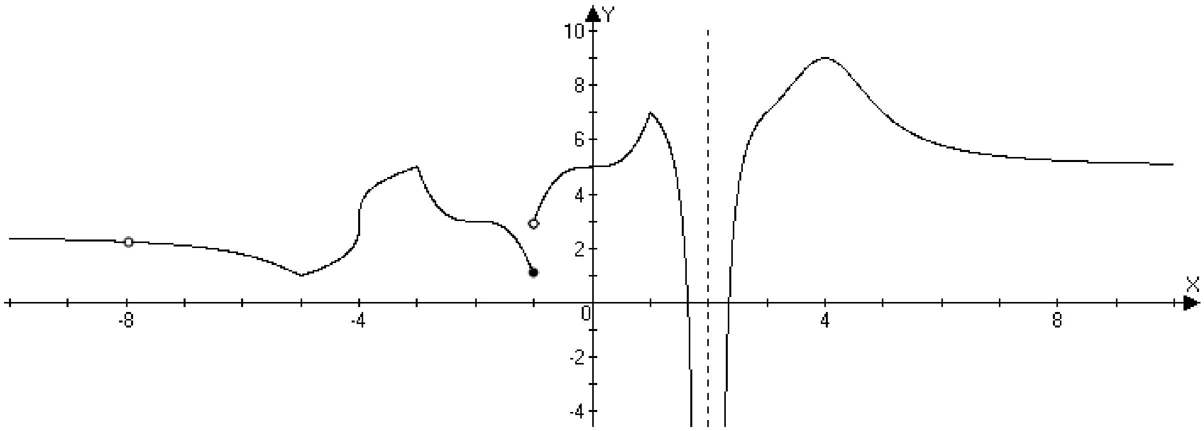
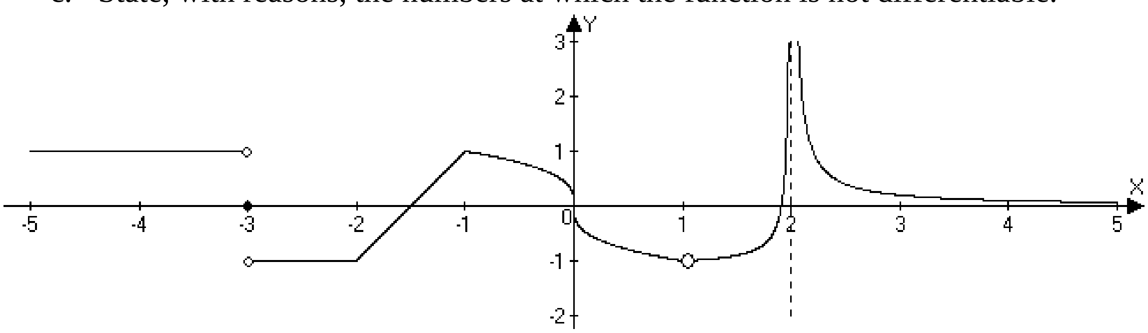
Math 1A – Final Exam Review Problems – fall '10

1. The graph below shows $y = f(x)$.
   a. State, with reasons, the numbers a at which $\lim_{x \to a} f(x)$ does not exist.
   b. State where the function is discontinuous and classify the discontinuity as removable, jump or asymptote.
   c. State, with reasons, the numbers at which the function is not differentiable.



2. The graph below shows $y = f(x)$.
   a. State, with reasons, the numbers a at which $\lim_{x \to a} f(x)$ does not exist.
   b. State where the function is discontinuous and classify the discontinuity as removable, jump or asymptote.
   c. State, with reasons, the numbers at which the function is not differentiable.



3. Suppose $f(x)$ is a function such that for all $x > 0$, $\frac{2}{\pi}\arctan x < f(x) < \frac{x+1}{x-1}$.

   What is $\lim_{x \to \infty} f(x)$? Why?

4. Compute the limit, if it exists.
   a. $\lim_{x \to 2} \frac{x-2}{x^2 - 3x + 2}$
   b. $\lim_{h \to 0} \frac{\sqrt{h+1} - 1}{h}$
   c. $\lim_{x \to \infty} \frac{2x^4 - x^2 + 4x}{16 - x^4}$
   d. $\lim_{\theta \to 0} \frac{\sin 5\theta}{2\theta}$

5. Use the definition of the derivative to derive one of the formula below (your choice)

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}} \quad \text{or} \quad \frac{d}{dx}(\sin x) = \cos x$$

6. Use the various shortcut rules (product rule, chain rule, etc.) to simplify the derivative:
   a. $f(t) = \dfrac{3t-5}{t^2+7t-11}$
   b. $g(x) = x^5 \tan(2x)$
   c. $h(u) = \sin(\sin(\sin u))$

7. Sketch a graph for a function defined on [-4,4] that meets the given conditions:
   $f(0)=1,\quad f'(-2)=f'(1)=f'(3)=0,\ f'(0)$ is undefined
   $f'(x)<0$ only on $(-2,0)\cup(1,3)$
   $f''(x)<0$ only on $(-4,0)\cup(0,2)$

8. Find a linear approximation for the function $R = \dfrac{100r}{100+r}$ near $a = 50$ and use it to approximate R when $r = 53$.

9. Find an equation for the line tangent to $y = e^{\cos 2x}$ at $x = \dfrac{\pi}{4}$.

10. Under what condition(s) will the points on a tangent line lie beneath the curve?

11. Find the local and global extreme values of the function $y = x^2 \ln x^2$ on the interval [-2,2].

12. Find the point on the parabola $x = 16 - y^2$ that is closest to the point (16, 3).

13. If $y = e^{xy}$, find an expression for $\dfrac{dy}{dx}$ and write a linearization for y near (0,1).

14. Find an equation for the line tangent to $x^3 + x^2y + 2y^3 = 2$ at $(2,-1)$.

15. Show that the y-coordinate of the point (x,y) on the curve described by $\dfrac{x^2+1}{\ln\left(y^2+4\right)} = 1$ that is closest to the point (0,2) can be found by solving $2y^3 - 4y^2 + 10y - 16 = 0$. Use Newton's method to solve the equation. *Hint*: Minimizing the distance D is equivalent to minimizing the square of the distance, D^2. Also, the curve has origin symmetry, so you can just work with the distance in the first quadrant.

16. A box with a square base with no top has a surface area of 108 square feet. Find the dimensions that will maximize the volume.

17. Use Newton's method to approximate a solution to $x^2 = \cos 2x$ to 8 digits. Simplify the iteration formula for this case, use an initial value of $x = 1$ and show what the iterates are.

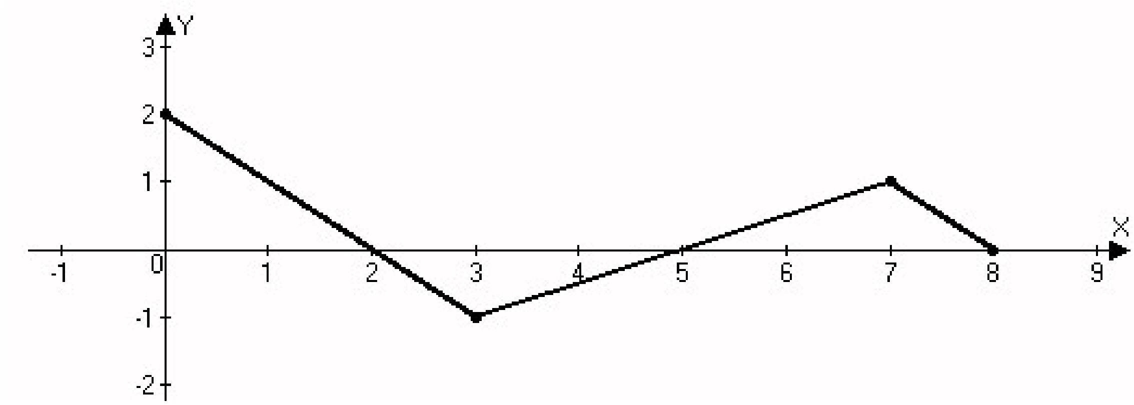
18. Sketch a graph for a function that meets the given conditions:

$f(0)=0, \quad f'(-4)=f'(0)=f'(4)=0$

$f'(x)<0$ on $(-\infty,-4)\cup(-4,0)\cup(4,\infty)$

$f''(x)>0$ on $(-\infty,-4)\cup(-2,2)\cup(6,\infty)$

19. The graph of $f'(x)$ is shown below.
   a. Sketch a graph for $f''(x)$.
   b. Sketch a possible graph for $f(x)$.



20. Find a linear approximation for the function $f(x)=\sin\left(x+\frac{\pi}{3}\right)$ at $a = 0$ and use it to approximate $f(-0.14)$.

21. Find the slope of the line tangent to the curve $x^2\cos y+\sin 2y = xy$ at $(x,y)=\left(0,\frac{\pi}{2}\right)$.

22. Find an equation for the line tangent to $y=e^{\sin 3x}$ at $x=\frac{\pi}{3}$.

23. At what point on the curve $y=\left[\ln(x+4)\right]^2$ is the tangent line horizontal?

24. Find the local and global extreme values of the function $y=x^3e^{-x^2}$ on the interval [-1,2].

25. Find the point on the hyperbola $xy = 16$ that is closest to the point (4,0).

26. Evaluate the limit: $\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{x}$.

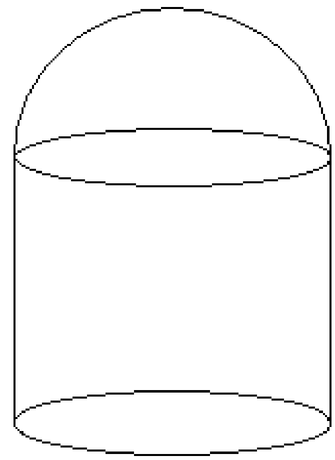
27. The velocity of a wave of length L in deep water is $v=k\sqrt{\frac{L}{C}+\frac{C}{L}}$ where k and C are known positive constants. What is the length of the wave that gives the minimum velocity?

28. What is the maximum slope of a line connecting the origin (0,0) with a point on the parabola

$y = 1 - (x-2)^2$?

29. Show that the *y*-coordinate of the point (*x*,*y*) on the curve described by $\dfrac{x^2+1}{\ln(y^2+4)} = 1$ that is closest to the point (0,2) can be found by solving $2y^3 - 4y^2 + 10y - 16 = 0$. Use Newton's method to solve the equation. *Hint*: Minimizing the distance *D* is equivalent to minimizing the square of the distance, D^2. Also, the curve has origin symmetry, so you can just work with the distance in the first quadrant.

30. A metal storage tank with volume *V* is to be constructed in the shape of a right circular cylinder with a hemispherical top. What radius and height will require the least amount of metal?



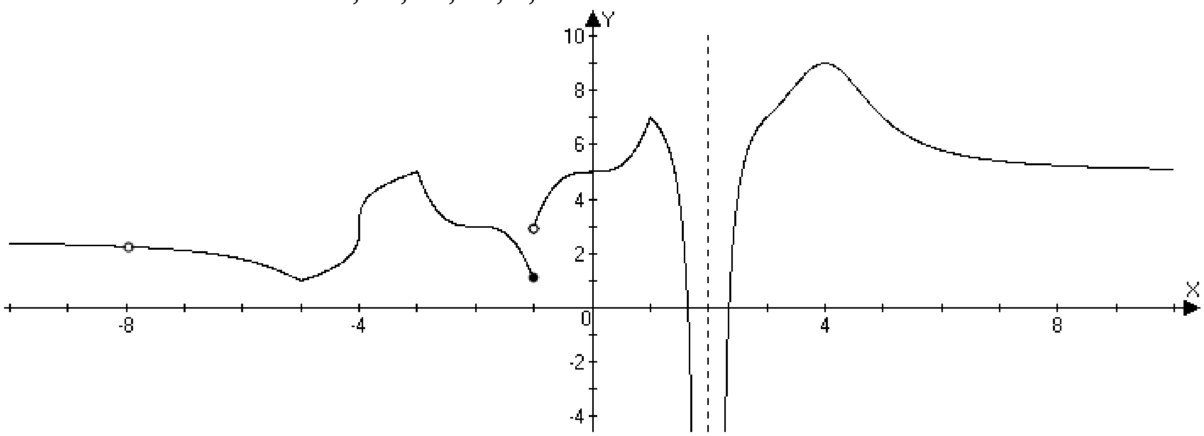
Math 1A – Final Exam Review Problems – Some Solutions – Fall '10

1. The graph below shows $y = f(x)$.
   a. State, with reasons, the numbers a at which $\lim_{x\to a} f(x)$ does not exist.

      SOLN: $\lim_{x\to -1} f(x)$ does not exist since there's a jump discontinuity from $y = 1$ to $y = 3$ there.

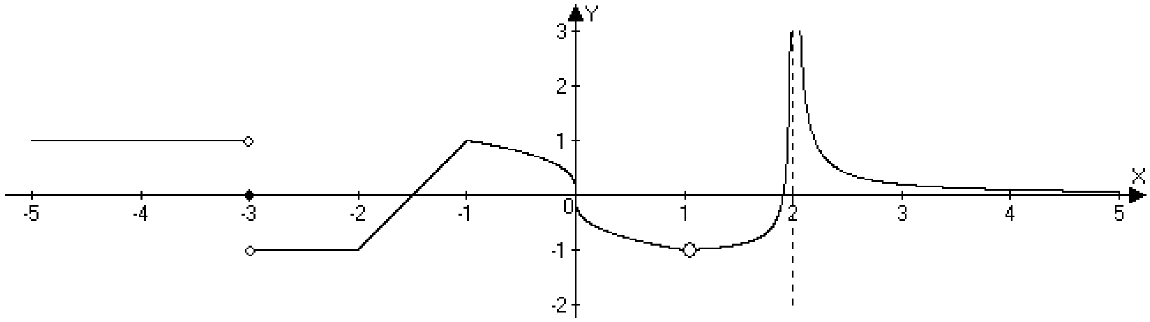
      Also, $\lim_{x\to 2} f(x)$ does not exist since there's a vertical asymptote there.

      In a nutshell: $x = -1$ and 2
   b. State where the function is discontinuous and classify the discontinuity as removable, jump or asymptote.
      SOLN: In addition to the two discontinuities from (a) above, there's a removable discontinuity at $x = –8$.
      In a nutshell: $x = –8, –1$ and 2
   c. State, with reasons, the numbers at which the function is not differentiable.
      SOLN: The function is not differentiable where it's not continuous: $x = –8, –1$ and 2. It also has corners (not smooth places) at $x = –5, –3$ and 1.
      Finally it's not differentiable at $x = –4$ since the tangent line is vertical there.
      In a nutshell: $x = –8, –5, –4, –3, 1, 2$



2. The graph below shows $y = f(x)$.
   a. State, with reasons, the numbers a at which $\lim_{x\to a} f(x)$ does not exist.

      SOLN: The limit does not exist at $a = -3$ since the limit from the left is 1 while the limit from the right is -1. Also, the limit does not exist at $a = 2$ since the value of f is unbounded above both from the left and the right.
   b. State where the function is discontinuous and classify the discontinuity as removable, jump or asymptote.
      SOLN: There is a jump discontinuity at $a = -3$, a removable discontinuity at $a = 1$ and a vertical asymptote discontinuity at $a = 2$.
   c. State, with reasons, the numbers at which the function is not differentiable.
      SOLN: The function is not differentiable where $a = -3$, 1 and 2, since these are discontinuities. The function is not differentiable at $a = -2$ somce the slope is 0 as approached from the left and 2 as approached from the right. The function is no differentiable at $a = -1$ since the slope is a constant 2 as approached from the left while the slope approaches some negative value from the right. The function is not differentiable at $a = 0$ since the tangent line is vertical there.



3. Suppose $f(x)$ is a function such that for all $x > 0$, $\frac{2}{\pi}\arctan x < f(x) < \frac{x+1}{x-1}$.

   What is $\lim_{x\to\infty} f(x)$? Why?

   SOLN: $\lim_{x\to\infty}\frac{2}{\pi}\arctan x = \frac{2}{\pi}\lim_{x\to\infty}\arctan x = \frac{2}{\pi}\frac{\pi}{2} = 1$ and $\lim_{x\to\infty}\frac{x+1}{x-1} = 1$ so, by the squeeze theorem, $\lim_{x\to\infty} f(x) = 1$

4. Compute the limit, if it exists.

   a. $\lim_{x\to 2}\frac{x-2}{x^2-3x+2} = \lim_{x\to 2}\frac{x-2}{(x-2)(x-1)} = \frac{1}{2-1} = 1$

   b. $\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}\left(\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}\right) = \lim_{h\to 0}\frac{h}{h\left(\sqrt{h+1}+1\right)} = \frac{1}{2}$

   c. $\lim_{x\to\infty}\frac{2x^4-x^2+4x}{16-x^4} = \lim_{x\to\infty}\frac{2-\frac{1}{x^2}+\frac{4}{x^3}}{\frac{16}{x^4}-1} = \frac{2-0+0}{0-1} = -2$

   d. $\lim_{\theta\to 0}\frac{\sin 5\theta}{2\theta} = \frac{5}{2}\lim_{5\theta\to 0}\frac{\sin 5\theta}{5\theta} = \frac{5}{2}$

5. Use the definition of the derivative to derive one of the formula below (your choice)

SOLN: $\frac{d}{dx}\left(\sqrt{x}\right) \equiv \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$

$$\frac{d}{dx}(\sin x) \equiv \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

$$= \sin x\lim_{h\to 0}\frac{(\cos h-1)}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h} = 0 + 1 * \cos x$$

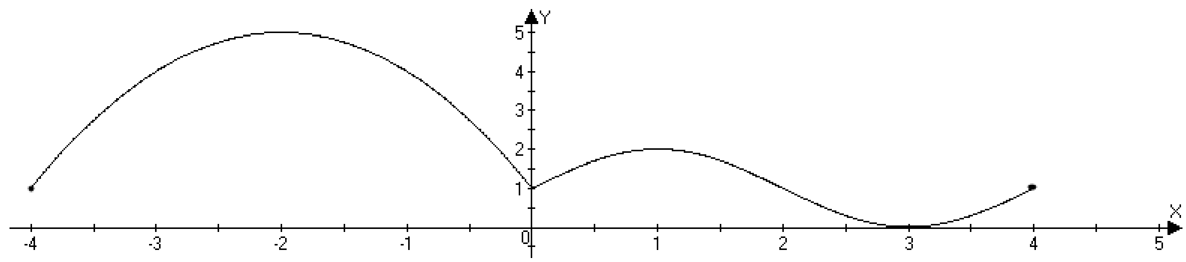
6. Use the various shortcut rules (product rule, chain rule, etc.) to simplify the derivative:

   a. $f(t) = \frac{3t-5}{t^2+7t-11} \Rightarrow f'(t) = \frac{\left(t^2+7t-11\right)3-(2t+7)(3t-5)}{\left(t^2+7t-11\right)^2} = \frac{-3t^2+10t+2}{\left(t^2+7t-11\right)^2}$

   b. $g(x) = x^5\tan(2x) \Rightarrow g'(x) = 5x^4\tan(2x) + 2x^5\sec^2(2x)$

   c. $h(u) = \sin(\sin(\sin u)) \Rightarrow h(u) = \cos(\sin(\sin u))\cos(\sin u)\cos u$

7. Sketch a graph for a function defined on [-4,4] that meets the given conditions:
$f(0)=1, \quad f'(-2)=f'(1)=f'(3)=0, \; f'(0)$ is undefined
$f'(x)<0$ only on $(-2,0)\cup(1,3)$
$f''(x)<0$ only on $(-4,0)\cup(0,2)$



8. Find a linear approximation for the function $R=\dfrac{100r}{100+r}$ near a = 50 and use it to approximate R when r = 53.
SOLN: $R'=\dfrac{(100+r)100-100r}{(100+r)^2}=\dfrac{10000}{(100+r)^2}$, So at a = 50, $R'=\dfrac{10000}{(150)^2}=\dfrac{4}{9}$. Thus the approximating line is $R=\dfrac{100}{3}+\dfrac{4}{9}(r-50)$. At r = 53, $R\approx\dfrac{100}{3}+\dfrac{4}{9}(53-50)=\dfrac{104}{3}$

9. Find an equation for the line tangent to $y=e^{\cos 2x}$ at $x=\dfrac{\pi}{4}$.
SOLN: $y'=-2\sin 2xe^{\cos 2x}$ so at $x=\dfrac{\pi}{4}$, the slope of the tangent line is $m=-2$ and the equation for the tangent line is $y=1-2\left(x-\dfrac{\pi}{4}\right)$.

10. Under what condition(s) will the points on a tangent line lie beneath the curve?
SOLN: If the curve is concave up, at least in the immediate neighborhood of the point.

11. Find the local and global extreme values of the function $y=x^2\ln x^2$ on the interval [-2,2].
SOLN: It may be a bit simpler to work with the equivalent formula, $y=2x^2\ln|x|$.
$y'=4x\ln|x|+2x=0$ If x is not 0 then $\ln|x|=-\dfrac{1}{2}\Leftrightarrow x=\pm e^{-1/2}\approx\pm 0.6065$

12. Find the point on the parabola $x=16-y^2$ that is closest to the point (16, 3).
SOLN: The square of the distance from (16, 3) to the parabola is
$D^2=(16-x)^2+(3-y)^2=\left(y^2\right)^2+(3-y)^2$. Differentiating with respect to y we have
$2D\dfrac{dD}{dy}=4y^3-2(3-y)=4y^3+2y-6=2(y-1)\left(2y^2+2y+3\right)$ so the only real value of y that assures $\dfrac{dD}{dy}=0$ is y = 1, which means x = 15.

13. If $y = e^{xy}$, find an expression for $\frac{dy}{dx}$ and write a linearization for *y* near (0,1).

SOLN: First write the equivalent logarithmic equation: $\ln y = xy$, the differentiate with respect to *x* implicitly: $\frac{1}{y}\frac{dy}{dx} = y + x\frac{dy}{dx}$. Plugging in (*x*, *y*) = (0, 1) we get $\frac{dy}{dx} = 1$. Thus the equation for the tangent line is *y* = *x* + 1.

14. Find an equation for the line tangent to $x^3 + x^2y + 2y^3 = 2$ at $(2,-1)$.

SOLN: Differentiating with respect to *x* implicitly: $3x^2 + 2xy + x^2\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 0$.

Plugging in the point coordinates, $12 - 4 + 4\frac{dy}{dx} + 6\frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{4}{5}$, so the tangent line is the solution set to $y = -1 - \frac{4}{5}(x-2) = -\frac{4}{5}x + \frac{3}{5}$

15. Show that the *y*-coordinate of the point (*x*,*y*) on the curve described by $\frac{x^2+1}{\ln\left(y^2+4\right)} = 1$ that is closest to the point (0,2) can be found by solving $y^3 - 2y^2 + 5y - 8 = 0$. Use Newton's method to solve the equation. *Hint*: Minimizing the distance *D* is equivalent to minimizing the square of the distance, D^2. Also, the curve has origin symmetry, so you can just work with the distance in the first quadrant.

SOLN: Here the square of the distance from (0,2) to the curve is $D^2 = (x-0)^2 + (y-2)^2 = x^2 + (y-2)^2$. On the curve, $x^2 = -1 + \ln\left(y^2+4\right)$. Substituting, for x^2, we get $D^2 = -1 + \ln\left(y^2+4\right) + (y-2)^2$. Differentiating with respect to *y*;

$2D\frac{dD}{dy} = \frac{2y}{y^2+4} + 2(y-2)$, and setting this to zero we have the equivalent equation

$y^3 - 2y^2 + 5y - 8 = 0$. To find a zero of this cubic, iterate

$$y_{n+1} = y_n - \frac{y_n^3 - 2y_n^2 + 5y_n - 8}{3y_n^2 - 4y_n + 5} = \frac{2y_n^3 - 2y_n^2 + 8}{3y_n^2 - 4y_n + 5}$$

The TI85 screen captures showing how this can be implemented are below (assuming this formula is in *y*4 on the graph page:

```
1→x
                        1
y4→x
                        2
             1.7777777778
             1.75246603387
             1.75217181803
```

```
1.7777777778
1.75246603387
1.75217181803
1.75217177888
1.75217177888
1.75217177888
1.75217177888
```

16. A box with a square base with no top has a surface area of 108 square feet. Find the dimensions that will maximize the volume.

SOLN: Let *x* = the side of the square base and let *y* = the height of the box. Then the surface area is $x^2 + 4xy = 108$ which means that the height of the box can be represented as

$y = \dfrac{108 - x^2}{4x}$. The volume of the box is $V = x^2y$. Substituting for y we have

$V = x^2 y = \dfrac{108x - x^3}{4} = 27x - \dfrac{x^3}{4}$. Thus the instantaneous rate of change in the volume per

change in x is $\dfrac{dV}{dx} = 27 - \dfrac{3x^2}{4} = 0 \Rightarrow x = 6 \Rightarrow y = 3$

17. Use Newton's method to approximate a solution to $x^2 = \cos 2x$ to 8 digits. Simplify the iteration formula for this case, use an initial value of $x = 1$ and show what the iterates are.

SOLN: $x^2 = \cos 2x \Leftrightarrow f(x) = x^2 - \cos 2x = 0$. Applying Newton's iteration formula we have

$x_{n+1} = x_n - \dfrac{x_n^2 - \cos 2x_n}{2x_n + 2\sin 2x_n}$. The iterates starting at $x = 1$ are then

```
                      1
y5→x
           .629144517598
           .601189561129
           .600769248799
            .60076914967
            .60076914967
```