Math 1A - §3.7–11 Test – Fall '10 Name________________________________

Show your work for credit. Write all responses on separate paper. Don't use a calculator.

1. Open cylindrical tubes resonate at the approximate frequencies, f, where

$$f = \frac{nv}{2(L+0.8d)}$$

Here n is a positive integer, v is the speed of sound in air (approximately 343 meters/sec on Earth near sea level) , L is the length of the tube (in meters) and d is the diameter of the resonance tube (meters).

   a. Find the rate of change of the frequency with respect to
      i. The length (when n and d and v are constant)
      ii. The diameter (when n and v and L are constant)
      iii. The velocity (when n and L and d are constant)

   b. For $d = 0.125$, $nv = 400$, use the differential df to approximate the change in frequency when the length of the tube increases from $L = 0.9$ to $L = 1.0$

   Recall that $\Delta f \approx df = f'(0.9)\Delta L$

2. The chart at right tabulates the populations (in millions) of Gabon and Gambia in the years 1960 and 1980.

	A	B	C
1	Population (millions)		
2	Year	Gabon	Gambia
3	1960	490	310
4	1980	686	620

Let $P_1(t)$ and $P_2(t)$ represent the populations Gabon and Gambia, respectively.

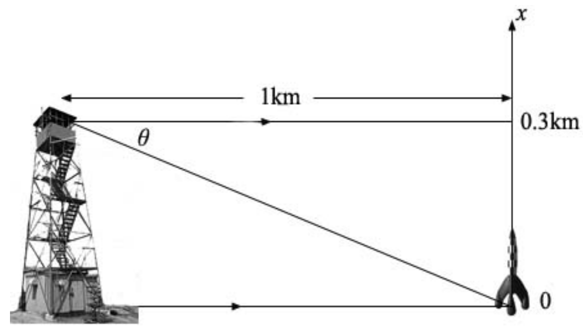
Assume that each population follows an exponential growth model: $P = P_0e^{rt}$, where $t = 0$ in the year 1960.

   a. Set up equation by evaluating the ratio, $\frac{P(20)}{P(0)}$, in terms of both the data and the formula.

   Then solve these to find the relative growth rates r_1 and r_2 for Gabon and Gambia populations, respectively. Simplify what you can without using a calculator (don't approx. the logarithms.)

   b. What equation would you solve to predict, based on this evidence, the time when the populations of Gabon and Gambia are equal? Don't solve the equation here, maybe later today.

3. A camera is positioned at the top of a tower of height 0.3 km. The base of the tower is 1km from the base of a rocket launching pad, as shown in the diagram. Assume the height (in km) of the rocket at time t (in sec) is given by $x(t) = 0.003t^2$ Let θ be the angle of elevation as shown in the diagram at right.



   a. Find an equation relating x and θ in terms of t.

   b. Find the rate of change of θ when $\theta = 0$.

4. The gas law for a fixed mass m (in moles) of an ideal gas at absolute temperature T (in Kelvins), pressure P (in atmospheres), and volume V (in liters) is

$$PV = mRT,$$

where $R \approx 0.82$ is the gas constant. Suppose that, at some instant, $P = 10$ atm and P is decreasing at a rate of 0.2 atm/sec, the volume is $V = 82$ L and V is increasing at a rate of 0.5 L/sec . Find the rate of change of the temperature at that instant if there are 100 moles of gas.

5. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a sphere with diameter 6 cm. Hint: The paint increases the volume of sphere slightly, and the formula for the volume V of a sphere in terms of its radius r is $V = \frac{4}{3}\pi r^3$.

6. Find a formula for the derivative function for $y = x^2 \sinh^{-1}(2x)$ Recall that $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$.

7. At what point along the curve $y = \cosh(x)$ does the tangent line have slope 1?

Math 1A - §3.7–11 Test Solutions – Fall '10.

1. Open cylindrical tubes resonate at the approximate frequencies, f, where

$$f=\frac{nv}{2\left(L+0.8d\right)}$$

Here n is a positive integer, v is the speed of sound in air (approximately 343 meters/sec on Earth near sea level) , L is the length of the tube (in meters) and d is the diameter of the resonance tube (meters).

a. Find the rate of change of the frequency with respect to

i. The length (when n and d and v are constant)

SOLN: $\frac{d}{dL}f=\frac{-nv}{2\left(L+0.8d\right)^2}$

ii. The diameter (when n and v and L are constant) (curious variable collision here…)

SOLN: $\frac{d}{dd}f=\frac{-0.4nv}{\left(L+0.8d\right)^2}$

iii. The velocity (when n and L and d are constant)

SOLN: $\frac{d}{dv}f=\frac{n}{2\left(L+0.8d\right)}$

b. For $d = 0.125$, $nv = 400$, use the differential df to approximate the change in frequency when the length of the tube increases from $L = 0.9$ to $L = 1.0$

Recall that $\Delta f \approx df = f'(0.9)\Delta L$

SOLN: $df=\frac{-400}{2\left(0.9+0.8\left(0.125\right)\right)^2}\left(0.1\right)=\frac{-40}{2\left(0.9+0.1\right)}=-20\,\text{Hz}$

2. The chart at right tabulates the populations (in millions) of Gabon and Gambia in the years 1960 and 1980.
Let $P_1(t)$ and $P_2(t)$ represent the populations Gabon and Gambia, respectively.
Assume that each population follows an exponential growth model: $P = P_0e^{rt}$, where $t = 0$ in the year 1960.

	A	B	C
1	Population (millions)		
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a. Set up equation by evaluating the ratio, $\frac{P(20)}{P(0)}$, in terms of both the data and the formula.

Then solve these to find the relative growth rates r_1 and r_2 for Gabon and Gambia populations, respectively. Simplify what you can without using a calculator (don't approx. the logarithms.)

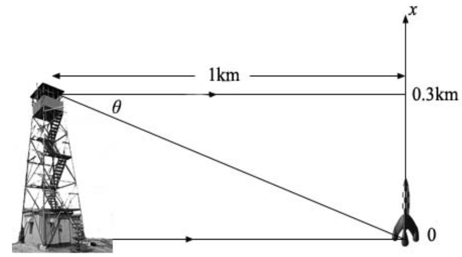
SOLN: For Gabon, $\frac{P(20)}{P(0)}=\frac{P_0e^{20r}}{P_0}=e^{20r}=\frac{686}{490}=1.4\Leftrightarrow r=\frac{\ln 1.4}{20}$

For Gambia, $\frac{P(20)}{P(0)}=\frac{P_0e^{20r}}{P_0}=e^{20r}=\frac{320}{160}=2\Leftrightarrow r=\frac{\ln 2}{20}$

b. What equation would you solve to predict, based on this evidence, the time when the populations of Gabon and Gambia are equal? Don't solve the equation here, maybe later today.

SOLN: $490e^{\ln 1.4t/20}=310e^{\ln 2t/20}\Leftrightarrow 490\left(1.4\right)^{t/20}=310\left(2\right)^{t/20}$

3. A camera is positioned at the top of a tower of height 0.3 km. The base of the tower is 1km from the base of a rocket launching pad, as shown in the diagram. Assume the height (in km) of the rocket at time t (in sec) is given by $x(t) = 0.003t^2$ Let θ be the angle of elevation as shown in the diagram at right.



a. Find an equation relating x and θ in terms of t.

SOLN: $\tan\theta = \dfrac{0.3 - x}{1}$

b. Find the rate of change of θ when $\theta = 0$.

SOLN: Find the rate of change of θ per change in x when $\theta = 0$.

$$\frac{d}{dx}\tan\theta = \frac{d}{dx}(0.3 - x) \Leftrightarrow \sec^2\theta\frac{d\theta}{dx} = -1 \Leftrightarrow \frac{d\theta}{dx} = -\cos^2\theta \text{, so at } \theta = 0, \boxed{\frac{d\theta}{dx} = -1}$$

Now this will occur when $t = 10$, at which time $\dfrac{dx}{dt} = 0.006t\Big|_{t=10} = 0.06$ so that

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = -1(0.06) = -0.06 \text{ rad/sec}$$

4. The gas law for a fixed mass m (in moles) of an ideal gas at absolute temperature T (in Kelvins), pressure P (in atmospheres), and volume V (in liters) is

$$PV = mRT,$$

where $R \approx 0.82$ is the gas constant. Suppose that, at some instant, $P = 10$ atm and P is decreasing at a rate of 0.2 atm/sec, the volume is $V = 82$ L and V is increasing at a rate of 0.5 L/sec . Find the rate of change of the temperature at that instant if there are 100 moles of gas.

SOLN: $\dfrac{d}{dt}PV = \dfrac{d}{dt}mRT \Leftrightarrow P'V + PV' = mRT'$. Now $P = 10$, $P' = -0.2$, $V = 82$ and $V' = 0.5$, so

$$-0.2(82) + 10(0.5) = 100(0.82)T' \Leftrightarrow -16.4 + 5 = 82T' \Leftrightarrow T' = \frac{-11.4}{82} = \frac{-57}{410} \text{ degrees } K \text{ per sec.}$$

5. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a sphere with diameter 6 cm. Hint: The paint increases the volume of sphere slightly, and the formula for the volume V of a sphere in terms of its radius r is $V = \dfrac{4}{3}\pi r^3$.

SOLN: The volume of paint is $\Delta V \approx dV = \dfrac{dV}{dr}\Delta r = 4\pi(3)^2(0.05) = 1.8\pi = \dfrac{9\pi}{5}\text{cm}^3$.

6. Find a formula for the derivative function for $y = x^2\sinh^{-1}(2x)$ Recall that $\dfrac{d}{dx}\sinh^{-1}x = \dfrac{1}{\sqrt{1+x^2}}$.

SOLN: $y' = 2x\sinh^{-1}(2x) + \dfrac{2x^2}{\sqrt{1+4x^2}}$

7. At what point along the curve $y = \cosh(x)$ does the tangent line have slope 1?

SOLN: $y' = \sinh(x) = \dfrac{e^x - e^{-x}}{2} = 1 \Leftrightarrow e^x - e^{-x} = 2 \Leftrightarrow e^{2x} - 2e^x = 1 \Leftrightarrow \left(e^x - 1\right)^2 = 2$

$\Leftarrow e^x - 1 = \sqrt{2} \Leftrightarrow x = \ln\left(1 + \sqrt{2}\right)$