Math 1A – §3.1-3.6 Test – Fall '10 Name____________________________________

Write all responses on separate paper. Show your work for credit. No calculators.

1. Find an equation of the tangent line to the curve at the given point.

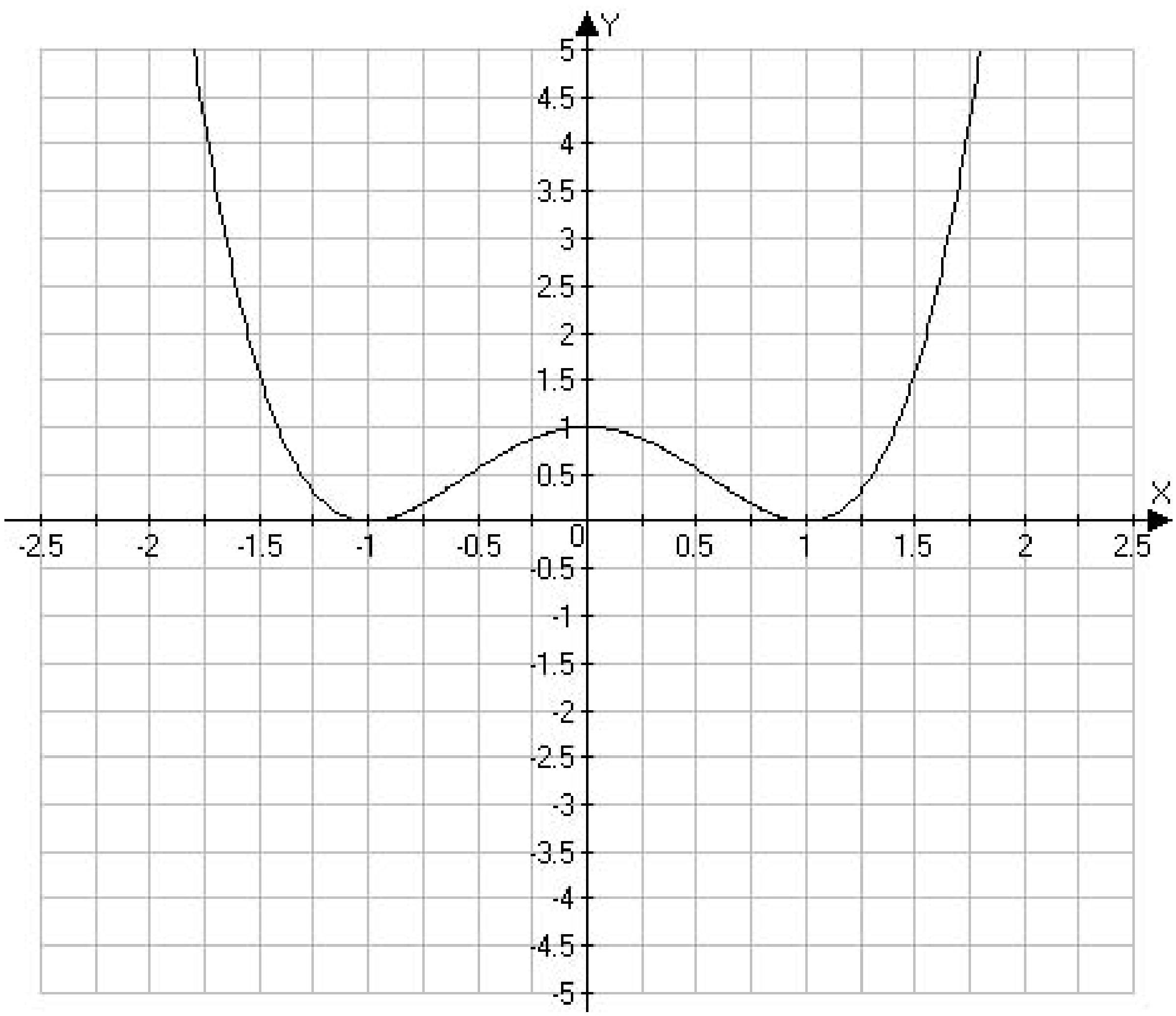
   a. $y=(2x-1)^3$ at (1,1).

   b. $y=\sin^2(\pi x)$ at $\left(\frac{1}{4},\frac{1}{2}\right)$

2. Let $f(x)=(x^2-1)^2$

   a. Find $f'(x)$ and $f''(x)$.

   b. The graph of $y=f(x)$ is shown below. Make tables, as necessary, and graph the functions $y=f'(x)$ and $y=f''(x)$ on top of the graph below:



3. Consider the function $f(x)=\dfrac{x+2}{x-1}$.

   a. Find a value of *x* where the slope of the tangent line to the curve is $-\dfrac{4}{3}$.

   b. Write an equation of that tangent line.

4. Prove, using the definition of the derivative, that if $f(x)=\cos(x)$, the $f'(x)=-\sin(x)$.

   You may use the facts that $\lim\limits_{u\to 0}\dfrac{\sin u}{u}=1$ and $\lim\limits_{u\to 0}\dfrac{1-\cos u}{u}=0$.

5. Find the first and second derivatives of $f(x)=e^{2x}\sin(3x)$.

6. Find the first and second derivatives of $y=\ln\left(x+\sqrt{x^2+1}\right)$

7. The displacement (in meters) of a particle on a string vibrating in simple harmonic motion is given by $s=a\cos(\omega t+\delta)$, where *t* is in seconds and *a*, ω, δ are constants. Find the following

   a. The velocity of the particle after *t* seconds.
   b. The maximum velocity, in terms of the parameters.

8. Find an equation of the tangent line to $x^2+4xy-4y^2+x=2$ at (1,0).

9. Find an equation of the tangent line to $y=x^{\sin(\pi x)}$ at (1,1).

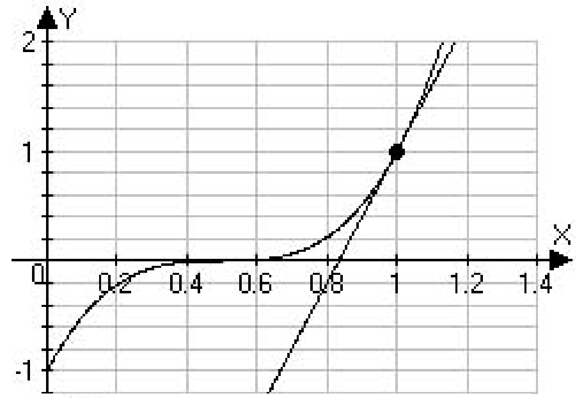
Math 1A – §3.1-3.6 Test Solutions – Fall '10.

1. Find an equation of the tangent line to the curve at the given point.

   a. $y=(2x-1)^3$ at (1,1).

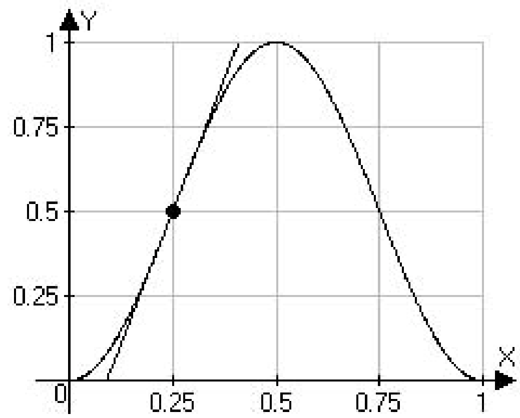
   SOLN: $y'=3(2x-1)^2(2)=6(2x-1)^2$ so the tangent line at (1,1) has slope $m=6(2-1)^2=6$. The point slope formula yields

   $$y-1=6(x-1)\Leftrightarrow y=6x-5$$



   b. $y=\sin^2(\pi x)$ at $\left(\frac{1}{4},\frac{1}{2}\right)$

   SOLN: You can differentiate directly:
   $y'=2\pi\sin(\pi x)\cos(\pi x)=\pi\sin(2\pi x)$
   so the tangent line where $x = ¼$ has slope
   $m=\pi\sin\left(2\pi\left(\frac{1}{4}\right)\right)=\pi\sin\left(\frac{\pi}{2}\right)=\pi$
   The point slope formula yields
   $y-\frac{1}{2}=\pi\left(x-\frac{1}{4}\right)\Leftrightarrow y=\pi x-\frac{\pi-2}{4}$

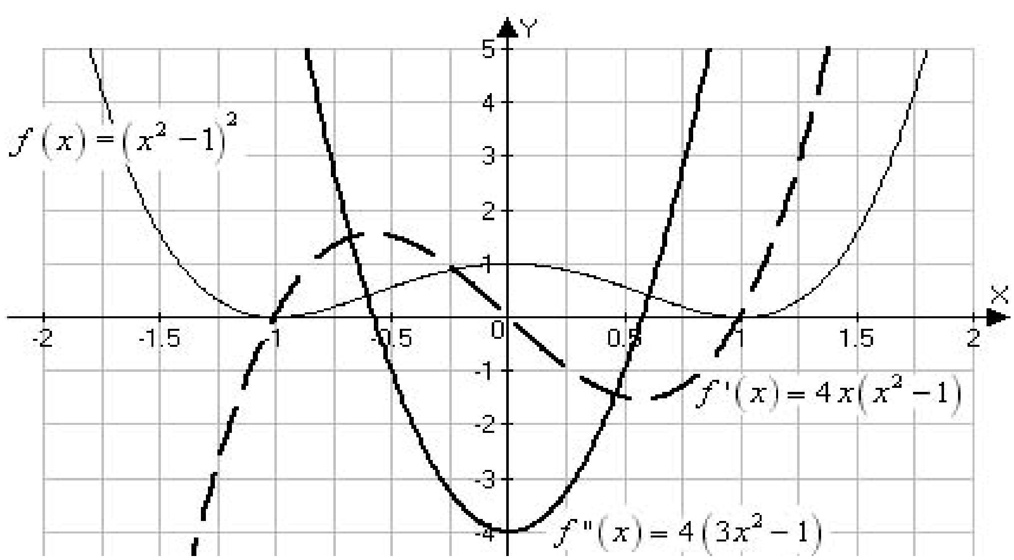


2. Let $f(x)=(x^2-1)^2$

   a. Find $f'(x)$ and $f''(x)$.

   SOLN: $f'(x)=2(x^2-1)(2x)=4x(x^2-1)$ and $f''(x)=4(x^2-1)+4x(2x)=4(3x^2-1)$

   b. The graph of $y=f(x)$ is shown below. Make tables, as necessary, and graph the functions $y=f'(x)$ and $y=f''(x)$ on top of the graph below:



3. Consider the function $f(x)=\dfrac{x+2}{x-1}$.

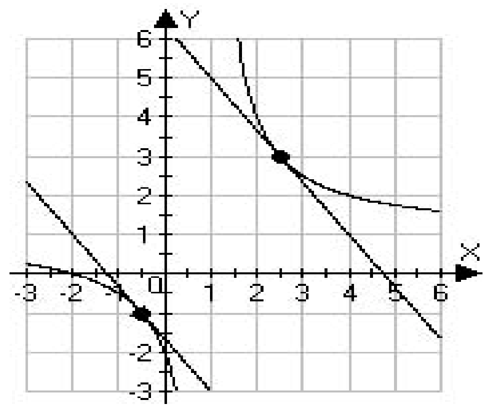
a. Find a value of x where the slope of the tangent line to the curve is $-\dfrac{4}{3}$.

SOLN: A little division algorithm will produce $f(x)=\dfrac{x+2}{x-1}=1+\dfrac{3}{x-1}$, which is slightly easier to differentiate: $f'(x)=\dfrac{-3}{(x-1)^2}$. So the slope the slope of the tangent line will be –4/3 when $\dfrac{-3}{(x-1)^2}=\dfrac{-4}{3}\Leftrightarrow (x-1)^2=\dfrac{9}{4}\Leftrightarrow x=1\pm\dfrac{3}{2}$. So either $x=\dfrac{5}{2}$ or $-\dfrac{1}{2}$.

b. Write an equation of that tangent line.

SOLN: $f\left(\dfrac{5}{2}\right)=1+\dfrac{3}{5/2-1}=1+2=3$ so the tangent line there is $y-3=-\dfrac{4}{3}\left(x-\dfrac{5}{2}\right)$

Also, $f\left(-\dfrac{1}{2}\right)=1+\dfrac{3}{-1/2-1}=1-2=-1$ so the tangent at (-0.5,-2) is $y+1=-\dfrac{4}{3}\left(x+\dfrac{1}{2}\right)$



4. Prove, using the definition of the derivative, that if $f(x)=\cos(x)$, the $f'(x)=-\sin(x)$.

You may use the facts that $\lim\limits_{u\to 0}\dfrac{\sin u}{u}=1$ and $\lim\limits_{u\to 0}\dfrac{1-\cos u}{u}=0$.

SOLN:
$$\frac{d}{dx}\cos x=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$
$$=\lim_{h\to 0}\frac{\cos x(\cos h-1)}{h}-\sin x\frac{\sin h}{h}=\cos x\lim_{h\to 0}\frac{(\cos h-1)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}=-\sin x$$

5. Find the first and second derivatives of $f(x)=e^{2x}\sin(3x)$. SOLN:

$$f'(x)=2e^{2x}\sin(3x)+3e^{2x}\cos(3x)=e^{2x}\left(2\sin(3x)+3\cos(3x)\right)=e^{2x}\sqrt{13}\sin\left(3x+\arctan\frac{3}{2}\right)$$
$$f''(x)=2e^{2x}\left(2\sin(3x)+3\cos(3x)\right)+e^{2x}\left(6\cos(3x)-9\sin(3x)\right)$$
$$=e^{2x}\left(12\cos(3x)-5\sin(3x)\right)=-13e^{2x}\sin\left(3x-\arctan\frac{12}{5}\right)$$

Note that

$$f''(x)=\frac{d}{dx}e^{2x}\sqrt{13}\sin\left(3x+\arctan\frac{3}{2}\right)=2e^{2x}\sqrt{13}\sin\left(3x+\arctan\frac{3}{2}\right)+3e^{2x}\sqrt{13}\cos\left(3x+\arctan\frac{3}{2}\right)$$
$$=e^{2x}\left(2\sqrt{13}\sin\left(3x+\arctan\frac{3}{2}\right)+3\sqrt{13}\cos\left(3x+\arctan\frac{3}{2}\right)\right)$$
$$=13e^{2x}\sin\left(3x+2\arctan\frac{3}{2}\right)=-13e^{2x}\sin\left(3x-\arctan\frac{12}{5}\right)$$

6. Find the first and second derivatives of $y=\ln\left(x+\sqrt{x^2+1}\right)$

SOLN: $y'=\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)=\frac{1}{x+\sqrt{x^2+1}}\left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)=\frac{1}{\sqrt{x^2+1}}$

Note that last simplification is a big one. If you missed it, then for *y*" you'd be doing this:

$$y''=\frac{-1}{\left(x+\sqrt{x^2+1}\right)^2}\left(1+\frac{x}{\sqrt{x^2+1}}\right)^2+\frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}-\frac{x^2}{\sqrt{x^2+1}}}{x^2+1}\right)$$

$$=\frac{1}{x+\sqrt{x^2+1}}\left(\frac{1}{\left(x^2+1\right)^{3/2}}\right)-\frac{1}{\left(x+\sqrt{x^2+1}\right)^2}\left(\frac{2x^2+1+2x\sqrt{x^2+1}}{x^2+1}\right)$$

$$=\frac{1}{\left(x+\sqrt{x^2+1}\right)\left(x^2+1\right)^{3/2}}-\frac{2x^2+1+2x\sqrt{x^2+1}}{\left(x+\sqrt{x^2+1}\right)^2\left(x^2+1\right)}$$

$$=\frac{x+\sqrt{x^2+1}-\left(2x^2+1+2x\sqrt{x^2+1}\right)\sqrt{x^2+1}}{\left(x+\sqrt{x^2+1}\right)^2\left(x^2+1\right)^{3/2}}$$

$$=\frac{x+\sqrt{x^2+1}-\left(2x^2+1\right)\sqrt{x^2+1}-2x\left(x^2+1\right)}{\left(x+\sqrt{x^2+1}\right)^2\left(x^2+1\right)^{3/2}}$$

$$=\frac{-x\left(2x\sqrt{x^2+1}+2x^2+1\right)}{\left(x+\sqrt{x^2+1}\right)^2\left(x^2+1\right)^{3/2}}=\boxed{\frac{-x}{\left(x^2+1\right)^{3/2}}}$$

Entering "second derivative of ln(x+sqrt(x^2+1)) into Wolfram Alpha yields the following

$$\frac{d^2}{dx^2}\left(\log\left(x+\sqrt{x^2+1}\right)\right)=\frac{\frac{1}{\sqrt{x^2+1}}-\frac{x^2}{\left(x^2+1\right)^{3/2}}}{\sqrt{x^2+1}+x}-\frac{\left(\frac{x}{\sqrt{x^2+1}}+1\right)^2}{\left(\sqrt{x^2+1}+x\right)^2}$$

This, it is said, has the "alternative form:" $-\frac{x}{\left(x^2+1\right)^{3/2}}$ Note the use of "log" for "ln".

7. The displacement (in meters) of a particle on a string vibrating in simple harmonic motion is given by $s=a\cos\left(\omega t+\delta\right)$, where *t* is in seconds and *a*, ω, δ are constants. Find the following
   a. The velocity of the particle after *t* seconds.

      SOLN: $v=\frac{ds}{dt}=-a\omega\sin\left(\omega t+\delta\right)$
   b. The maximum velocity, in terms of the parameters.

      SOLN: The maximum velocity is going to be $|a\omega|$

8. Find an equation of the tangent line to $x^2+4xy-4y^2+x=2$ at (1,0).

SOLN: $$2x+4y+4x\frac{dy}{dx}-8y\frac{dy}{dx}+1=0$$

$$(8y-4x)\frac{dy}{dx}=2x+4y+1$$

Substituting (x, y) = (1,0) we get $(-4)\frac{dy}{dx}=2+1 \Leftrightarrow \frac{dy}{dx}=-\frac{3}{4}$ so the equation of the tangent line is

$$\boxed{y=-\frac{3}{4}(x-1)}$$

9. Find an equation of the tangent line to $y=x^{\sin(\pi x)}$ at (1,1).

SOLN:

$$\ln y=\sin(\pi x)\ln x \Rightarrow \frac{1}{y}\frac{dy}{dx}=\pi\cos(\pi x)\ln x+\frac{\sin(\pi x)}{x} \Rightarrow \frac{dy}{dx}=\pi y\cos(\pi x)\ln x+\frac{y\sin(\pi x)}{x}$$

You could go further to solve for *dy/dx* in terms of *x*, but it's simpler to just substitute (x,y) = (1,1) here:

$\frac{dy}{dx}=\pi\cos(\pi)\ln 1+\frac{\sin(\pi)}{1}=0+0$ so the equation of the tangent line is $y = 1$.