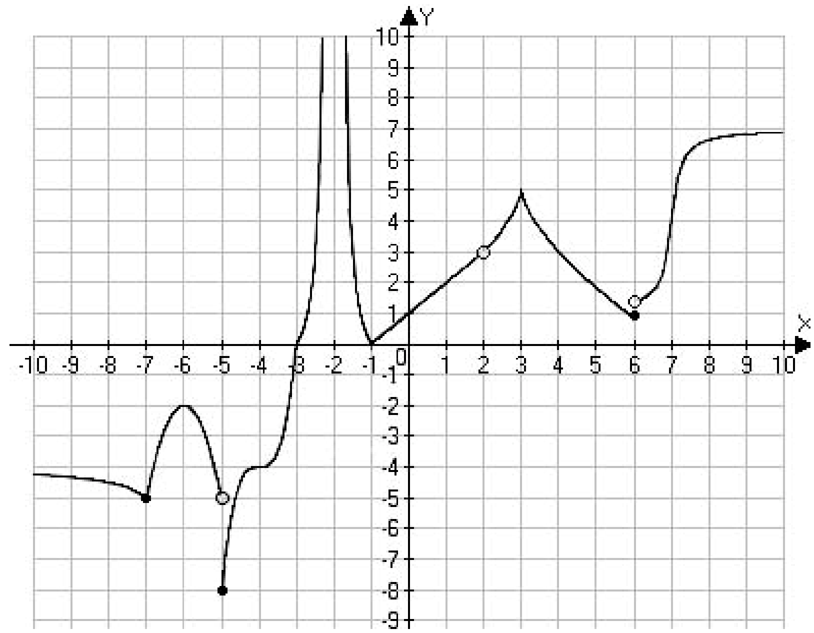
Math 1A – Chapter 2 Test – Fall ’10   Name:________________________________

Show your work for credit. Write all responses on separate paper. Do not use a calculator.

1. The graph of $y = f(x)$ is given.



a. Find $\lim_{x \to -7} f(x)$ if it exists, or explain why it doesn’t exist.

b. Find $\lim_{x \to -5^-} f(x)$ if it exists, or explain why it doesn’t exist.

c. Find $\lim_{x \to -5^+} f(x)$ if it exists, or explain why it doesn’t exist.

d. Find $\lim_{x \to 2} f(x)$ if it exists, or explain why it doesn’t exist.

e. For what value(s) of x does the function have a removable discontinuity?

f. For what value(s) of x does the function have a jump discontinuity?

g. What horizontal and vertical asymptote(s) does the graph suggest?

h. Where does the derivative function $f'(x)$ have a jump discontinuity?

2. Find the limit.

a. $\lim_{x \to 4} \dfrac{x^2 - 16}{x^3 - 64}$ b. $\lim_{x \to \infty} \dfrac{x^2 - 3}{\sqrt{9x^4 - x^2}}$ c. $\lim_{x \to \infty} \dfrac{1}{1 + e^{-x}}$ d. $\lim_{x \to 1} \dfrac{1}{\ln\left|x^2 - 1\right|}$

3. Prove each statement using the precise definition of the limit.

a. $\lim_{x \to 1} (3x - 1) = 2$

b. $\lim_{x \to 1} \dfrac{1}{\sqrt{x - 1}} = \infty$

4. Use the intermediate value theorem to prove that $2x^3 + x^2 + 2 = 0$ has a solution in $(-2, -1)$. Be sure to carefully indicate that the conditions of the theorem are satisfied for some function

5. If the tangent to $y = f(x)$ at (5,3) passes through the point (1,2), find $f'(5)$.

6. Find the derivative function for $f(x) = x^3 + x$ using the definition of the derivative.

7. Is there a number a such that $\lim_{x \to 1} \dfrac{x + 2a}{x^2 + x - 2}$ exists? If not, why not? If so, find the value of a and the value of the limit.

8. Consider $\lim_{x\to 0}\sin\left(x+e^x\right)$.

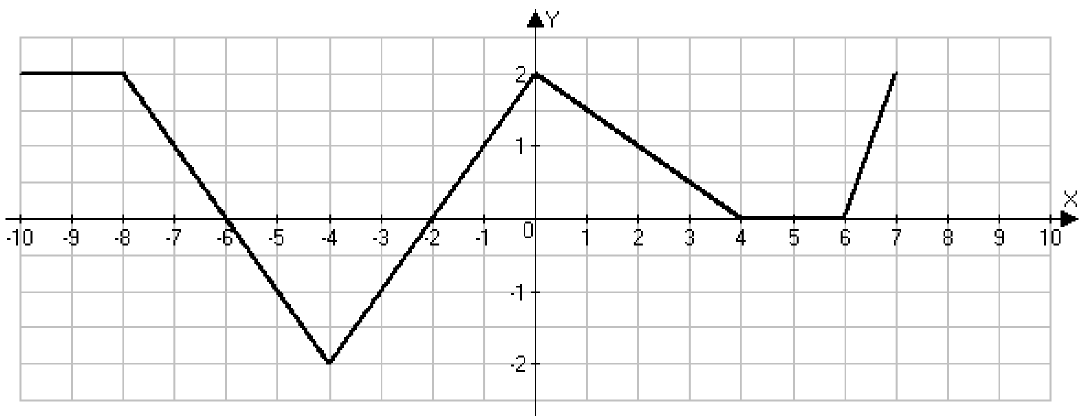
a. State a theorem that is needed to evaluating this limit. Why are the conditions of the theorem met?
b. Use the theorem to evaluate the limit.

9. Consider $\lim_{x\to 0}\frac{\sin(x)}{x}$ and assume you've established the inequality $\cos x \le \frac{\sin x}{x} \le 1$ for x near zero.

a. State a theorem that is useful to evaluating this limit. Why are the conditions of the theorem met?
b. Use the theorem to evaluate the limit.

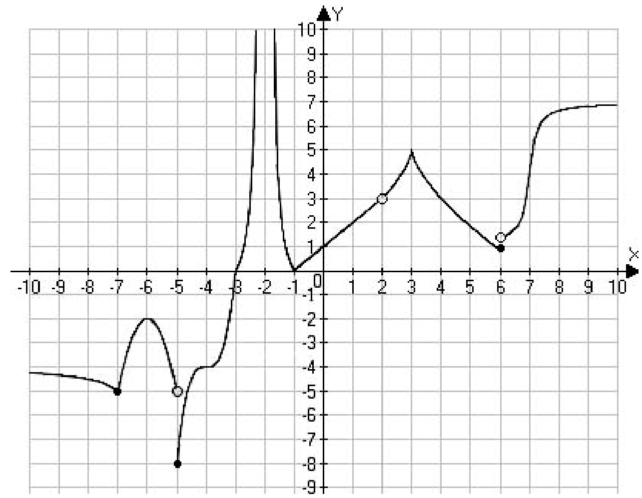
10. For the function $f(x)$ whose derivative function $f'(x)$ is graphed below, find where:

a. $f(x)$ is increasing
b. $f(x)$ has a local maximum.
c. $f''(x)$ is positive.
d. $f''(x)=0$.



Math 1A – Chapter 2 Test Solutions – Fall ’10

1. The graph of $y = f(x)$ is given.



a. $\lim_{x\to -7} f(x) = -5$.

b. $\lim_{x\to -5^-} f(x) = -5$

c. $\lim_{x\to -5^+} f(x) = -8$.

d. $\lim_{x\to 2} f(x) = 3$.

e. There is a removable discontinuity where $x = 2$.

f. There are jump discontinuities where $x = -5$ and where $x = 6$.

g. The horizontal asymptotes appear to run along $y = -4$ and $y = 7$. The vertical asymptote appears to be along $x = -2$.

h. The derivative function $f'(x)$ has jump discontinuities where the slope changes instantaneously from one value to another. This occurs at (–7, –5), (–3,0), (–1,0), at (3,5), where $x = -5$ and where $x = 6$.

2. Find the limit.

a. $\lim_{x\to 4}\frac{x^2-16}{x^3-64} = \lim_{x\to 4}\frac{(x-4)(x+4)}{(x-4)(x^2+4x+16)} = \lim_{x\to 4}\frac{x+4}{x^2+4x+16} = \frac{1}{8}$

b. $\lim_{x\to\infty}\frac{x^2-3}{\sqrt{9x^4-x^2}}\,\frac{\div x^2}{\div\sqrt{x^4}} = \lim_{x\to\infty}\frac{1-\frac{3}{x^2}}{\sqrt{9-\frac{1}{x^2}}} = \frac{1}{3}$

c. $\lim_{x\to\infty}\frac{1}{1+e^{-x}} = \frac{1}{1+\lim_{x\to\infty} e^{-x}} = \frac{1}{1+0} = 1$

d. $\lim_{x\to 1}\frac{1}{\ln\left|x^2-1\right|} = \frac{1}{\lim_{x\to 1}\ln\left|x^2-1\right|} = \frac{1}{-\infty} = 0$

3. Prove each statement using the precise definition of the limit.

a. $\lim_{x\to 1}(3x-1) = 2$ PROOF: Given any $\varepsilon > 0$, Choose $\delta = \frac{\varepsilon}{3}$ then

$$|x-1| < \delta \Rightarrow |x-1| < \frac{\varepsilon}{3} \Rightarrow 3|x-1| < \varepsilon \Rightarrow |3x-3| < \varepsilon \Rightarrow |f(x)-2| < \varepsilon$$

b. $\lim_{x\to 1}\frac{1}{\sqrt{x-1}} = \infty$ PROOF: Let N be a number as large as is required. Let $\delta < \frac{1}{N^2}$ then

$|x-1| < \delta \Rightarrow |x-1| < \frac{1}{N^2} \Rightarrow \frac{1}{|x-1|} > N^2 \Rightarrow \frac{1}{\sqrt{x-1}} > N$. Actually, an exacting pedant may object to the last step in this sequence of implications. Why?

4. Use the intermediate value theorem to prove that $2x^3 + x^2 + 2 = 0$ has a solution in $(-2,-1)$.

SOLN: $f(x) = 2x^3 + x^2 + 2$ is a polynomial function, so we know it's continuous everywhere.

$f(-2) = 2(-2)^3 + (-2)^2 + 2 = 2(-8) + 4 + 2 = -10$ and $f(-1) = 2(-1)^3 + (-1)^2 + 2 = 2(-1) + 1 + 2 = 1$

Thus 0 is a number between $f(-2)$ and $f(-1)$ and by the IVT, there is a value c between -2 and -1 such that $f(c) = 0$. This number c is then the required solution to the equation $2x^3 + x^2 + 2 = 0$.

5. If the tangent to $y = f(x)$ at (5,3) passes through the point (1,2), then $f'(5) = \frac{\Delta y}{\Delta x} = \frac{3-2}{5-1} = \frac{1}{4}$.

6. Find the derivative function for $f(x) = x^3 + x$ using the definition of the derivative.

SOLN: $f'(x) = \lim_{a \to x} \frac{f(x) - f(a)}{x-a} = \lim_{a \to x} \frac{x^3 + x - a^3 - a}{x-a} = \lim_{a \to x} \frac{(x-a)(x^2 + ax + a^2) + (x-a)}{x-a}$

$= \lim_{a \to x} x^2 + ax + a^2 + 1 = 3x^2 + 1$

7. Is there a number a such that $\lim_{x \to 1} \frac{x+2a}{x^2+x-2}$ exists? If not, why not? If so, find the value of a and the value of the limit. SOLN:

$\lim_{x \to 1} \frac{x+2a}{x^2+x-2} = \lim_{x \to 1} \frac{x+2a}{(x+2)(x-1)}$ so if $a = -½$ then the discontinuity is removable and the limit is 1/3.

8. Consider $\lim_{x \to 0} \sin(x + e^x)$.

a. Theorem: if $L = \lim_{x \to a} f(x)$ exists and g is continuous at L then $\lim_{x \to a} g[f(x)] = g\left[\lim_{x \to a} f(x)\right] = g(L)$
In this case, all the functions involved are continuous everywhere, so the conditions of the theorem are met.

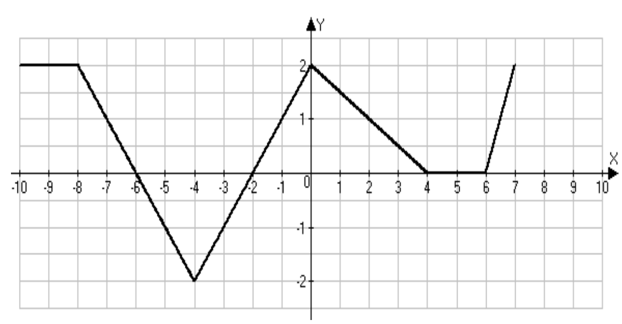
b. $\lim_{x \to 0} \sin(x + e^x) = \sin\left(\lim_{x \to 0} x + \lim_{x \to 0} e^x\right) = \sin(1)$.

9. Consider $\lim_{x \to 0} \frac{\sin(x)}{x}$ and assume you've established the inequality $\cos x \le \frac{\sin x}{x} \le 1$ for x near zero.

a. The squeeze theorem says that if $f(x) \le g(x) \le h(x)$ in some neighborhood of a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$. In this case, $1 = \lim_{x \to 0} 1 = \lim_{x \to 0} \cos(x)$

b. Thus $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

10. For the function $f(x)$ whose derivative function $f'(x)$ is graphed at right,



a. f is increasing on $(-10,-6) \cup (-2,4) \cup (6,7)$

b. f has a local maximum where x = -6

c. $f''(x)$ is positive on $(-4,0) \cup (6,7)$

d. $f''(x) = 0$ on $(-10,-8) \cup (4,6)$.