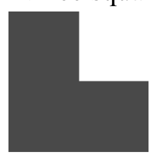
Math 15 - Chapters 3 and 4 Test

Show your work for each problem. Give thorough explanations in each case, using careful exposition to construct your answers. You may consult with others, but all the writing and understanding must be your own.

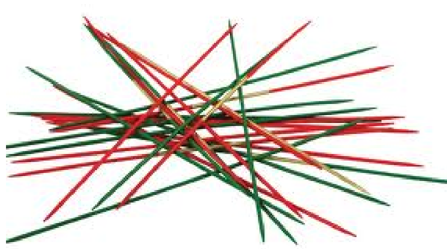
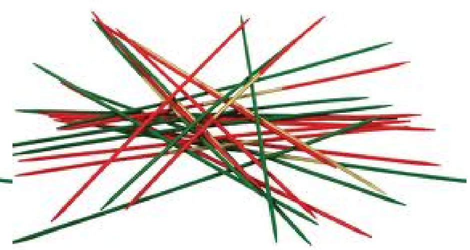
1. Let $C(n)$ be the coefficient of x^3 in the expansion of $(2x+3)^n$.
   Prove by induction on *n* that $C(n) = \binom{n}{3} 2^n 3^{n-3}$.

2. Let *n* be a positive integer. And let $P(n)$ be the statement that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using an el-shaped tile with three squares:

   Use induction to Prove $P(n)$. *Note: the removed square could be anywhere on the checkerboard.*

3. Two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game.

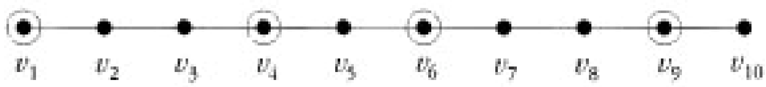
   
   

   Use induction to prove that if the two piles initially each contain the same number of sticks, the second player can always guarantee a win.

4. What is wrong with the following proof that every set of lines in the plane, no two of which are parallel, meet in a common point?
   Base case: $P(2)$ is true by the definition of parallel lines.
   Inductive hypothesis: Assume $P(k)$ is true, that is, every set of *k* lines meet in a common point.
   Inductive step: Consider a set of *k* + 1 line in the plane, no two of which are parallel. By the inductive hypotheses, the first *k* of them meet in a point, *p*. Also by the inductive hypothesis, the last *k* of these lines meet in a point *q*. If *p* and *q* were different points, then all the lines that contain both of them would be equal, a contradiction. Therefore, *p* = *q* and all the lines meet at a single point.

5. Let F_n be the *n*th Fibonacci number and let L_n be the *n*th Lucas number. Prove the following:
   a. $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$
   b. $L_{m+n} = F_{m+1}L_n + F_mL_{n-1}$
   c. $F_{n-1} + F_{n+1} = L_n$
   d. $L_{n-1} + L_{n+1} = 5F_n$
   e. $F_nL_n = F_{2n}$
   f. $F_n^2 = F_{n-1}F_{n+1} + (-1)^{n-1}$

6. The following problems are related.
   a. Let a_n denote the number of length *n* binary sequence with no consecutive 0's. For instance, 0110101101. Show that a_n satisfies the recurrence relation, $a_n = a_{n-1} + a_{n-2}$.
   b. Let b_n denote the number of ways to select a subset of nonadjacent vertices from a path on *n* vertices (as in the figure below, where nonadjacent vertices v_1, v_4, v_6, v_9 are chosen.) Such a subset of vertices is called an *independent set*. Notice that b_{n-2} and b_{n-1} count the number of independent subsets that <u>do</u> and <u>do not</u> contain the first point on the path, <u>respectively</u>. Show that $b_n = F_{n+2}$, a Fibonacci number.

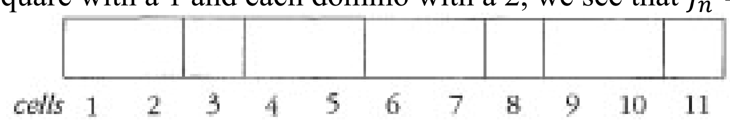


   This establishes a correspondence between independent sets of vertices and the 0's of binary sequences. Note that 0110101101 from part (a) corresponds to the independent set above.
   c. Note that there is a natural correspondence between these two representations, a_n and b_n: independent sets of vertices correspond to 0's in in the binary sequences.

   Let c_n denote the number of series of 1's and 2's that add to *n*. Then $c_1 = 1$ and $c_2 = 2$ since 1 = 1 is the only way to represent 1 and 2 = 2 = 1+1 are the two ways to represent 2. Prove that $c_n = F_{n+1}$.

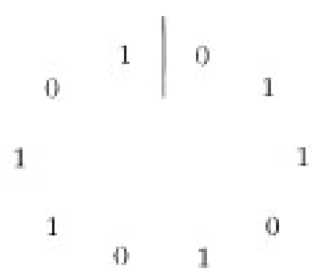
   Note that there is a natural correspondence between c_{n+1} and b_n. For a given series of 1's and 2's that add to *n* + 1, associate the subset of vertices whose *indices* are not partial sums of the series. For example, the series 2 + 1 + 2 + 2 + 1 + 2 + 1 = 11 has partial sums 2, 3, 5, 7, 8, 10 and 11 yielding the independent set v_1, v_4, v_6, v_9.

   d. Now consider the number of ways to tile a $1 \times n$ checkerboard with cells labeled $1, 2, \ldots, n$. Let f_n denote the number of ways to tile an *n*-board with 1×1 *squares* and 1×2 *dominoes*. Associating each square with a 1 and each domino with a 2, we see that $f_n = c_n$.



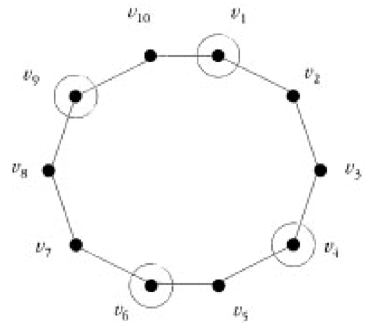
   Thus, $f_n = F_{n+1}$. So f_{m+n} is the number of ways to tile a length $m + n$ board. Explain why $f_{m+n} = f_m f_n + f_{m-1} f_{(n-1)}$.

7. Lucas numbers act like Fibonacci numbers running in circles.
   a. Let A_n denote the number of length *n circular* binary sequences with no consecutive 0's (as in the figure.) What are the length 2 and 3 circular sequences?

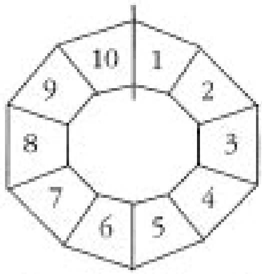


   b. Prove that $A_n = L_n$, the *n*th Lucas number.

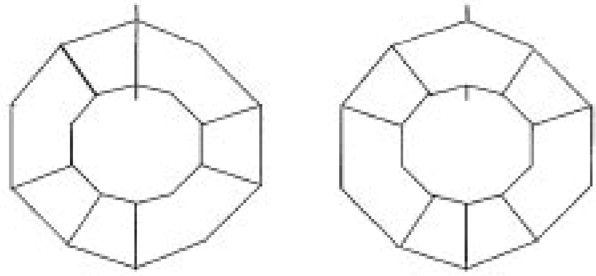
c. Let $B_n = A_n = L_n$ denote the number of independent sets in a cycle graph with n vertices.
Let C_n denote the number of series of 1's and 2's that sum to n with the end point restriction that it may not begin and end with a 2. Show that $C_n = L_{n-1}$.



d. Let l_n denote the number of ways to tile a circular $1 \times n$ board with squares and dominoes. Cells are labeled 1 through n and a tiling is called an n-bracelet. (see below.) An n-bracelet is *out of phase* if a domino covers cells n and 1, otherwise the n-bracelet is *in phase*. Show that the number of in phase n-bracelets is $f_n = F_{n+1}$ and the number of out of phase n-bracelets is $f_{n-2} = F_{n-1}$ and that $l_n = L_n$.



Circular 10-board



In phase

Out of phase

8. Consider the following function p, where L is a list.
   B. If $L = x$, a single element, then $p(L)$ = "x".
   R. If $L = L', x$ for some list L', then $p(L)$ = "x, $p(L')$".
   If L = `john, paul, george, ringo,` what is $p(L)$?

9. Suppose L is an SList of depth p. Find a recurrence relation for $A(p)$, the number of times two numbers are added when evaluating Sum(L).

10. Let L be an SList. Define a recursive function Wham as follows.
    B. Suppose $L = x$. Then Wham(L) = $x \cdot x$.
    R. Suppose $L = (X, Y)$. Then
    Wham(L) = Wham(X) + Wham(Y).
    a. Evaluate $\text{Wham}\big((2,4)(6,7)\big)$. Remember to show all work.
    b. Give a recurrence relation for $S(p)$, the number of + operations performed by Wham on an SList of depth p, for $p \geq 0$.
    c. Give a recurrence relation for $M(p)$, the number of $\cdot$ operations performed by Wham on an SList of depth p, for $p \geq 0$.D

11. An urn contains six red balls, six white balls, and six blue balls, and sample of three balls is drawn at random without replacement. Compute the probability that the sample contains at least one ball of each color. (Round your answer to four decimal places.)

12. An urn contains two red balls and five blue balls. Draw two balls at random from the urn, without replacement. Compute the expected number of red balls in your sample. (Round your answer to four decimal places.)

13. Consider the following algorithm.

```
for i ∈ {1, 2, 3, 4} do
  ┌ beep
    For j ∈ {1, 2, 3} do
      beep
    for k ∈ {1, 2, 3, 4} do
      ┌ for l ∈ {1, 2, 3, 4, 5, 6} do
          beep
        for m ∈ {1, 2, 3, 4, 5} do
  └   └   beep
```

How many times does a **beep** statement get executed?

14. Let $x_1, x_2, \ldots, x_n$ be an array. Consider the following algorithm.

```
for i ∈ {1, 2, ... , ⌊n/2⌋} do
  ┌ t ← x_i
    x_i ← x_{n−i+1}
  └ x_{n−i+1} ← t
```

a. How many ⟵ operations does this algorithm perform? Your answer should be a function of *n*.
b. What does this algorithm do to the array?

15. An urn contains *m* red balls and *n* green balls.

a. Give a big-Θ estimate for the number of ways to draw a sequence of *n* green balls without replacement.
b. Give a big-Θ estimate for the number of ways to choose 2 red balls and 3 green balls (assuming $m > 1$ and $n > 2$) without replacement.