Math 15 - Chapters 1 and 2 Test

Do any 14 of the following 20 problems. Due Monday, 3/19/12.

1. Consider the declaratives statements,
   P: The temperature outside is over 50°C
   Q: John will fry an egg
   R: Mary is hungry
   Using logical connectives, write a proposition which symbolizes the following:
   (a) If it is over 50°C and Mary is hungry, then John will fry an egg on the side walk.
   (b) John will fry an egg on the side walk only if the temperature outside is over 50°C
   (c) The temperature outside is not over 50°C.
   (d) If John does not fry an egg on the side walk, then either Mary is not hungry or it is not over 50°C.

2. Determine which of the following statements is true.
   _ Exactly one of these statements is false.
   _ Exactly two of these statements are false.
   _ Exactly three of these statements are false.
   _ Exactly four of these statements are false.
   _ Exactly five of these statements are false.
   _ Exactly six of these statements are false.
   _ Exactly seven of these statements are false.
   _ Exactly eight of these statements are false.
   _ Exactly nine of these statements are false.
   _ Exactly ten of these statements are false.

3. Construct a truth table for the proposition $P = [p \rightarrow (q \vee r)] \wedge [\neg(p \leftrightarrow \neg r)]$.

4. Is $\neg R \wedge (T \leftrightarrow P \leftrightarrow T \vee L)P$ a well formed expression? If not, how could you introduce new symbols into the express to make it a well-formed expression?

5. Suppose we know A is true, $A \rightarrow (B \rightarrow C)$ is true and that $B \rightarrow D$ is true. Can we conclude that $\neg C \rightarrow D$? Explain…or, that is, prove your claim.

6. Given that $\neg A \rightarrow B$ and $B \rightarrow A$ and $A \rightarrow \neg B$ can we conclude $\wedge \neg B$? Use the method of indirect proof to prove or disprove.

7. Prove that implication is not associative.

8. What are the axioms of the propositional logic?/-*

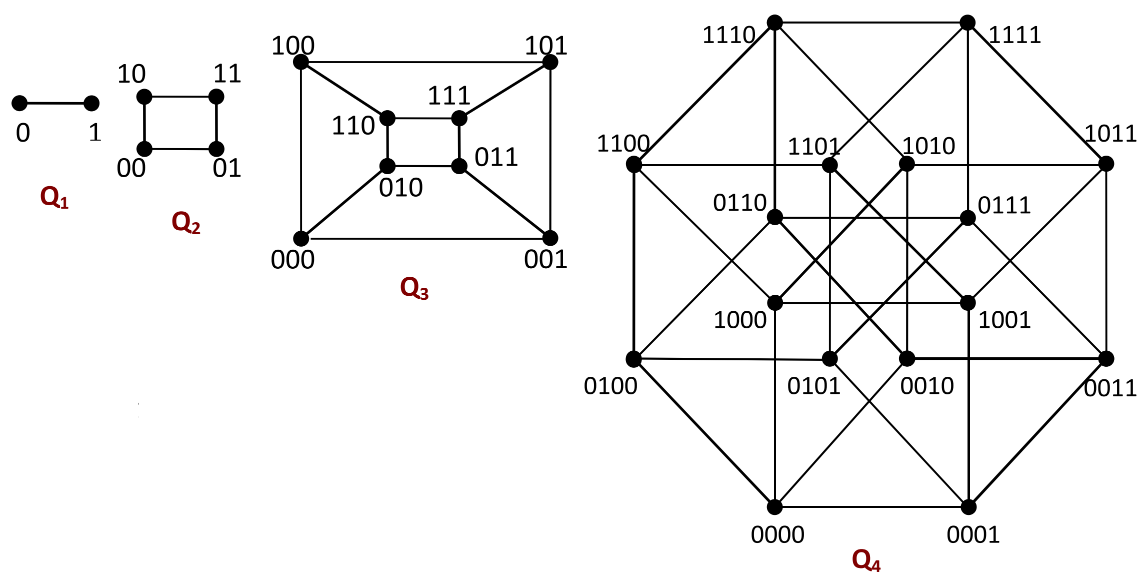
9. Prove or disprove: $(A \rightarrow B) \vee (A \rightarrow C) \rightarrow B \vee C$ is equivalent to $A \vee B \vee C$.

10. First, translate the following argument into propositional logic. Then, prove that the argument is valid using the method of formal derivation. Explain your answer.
Sally is destined to be either a fearless adventurer or a great psychiatrist. If Robert is not paranoid then Sally is not destined to be a great psychiatrist. Yet Sally is clearly not destined to be fearless adventurer. So Robert is definitely paranoid.

11. The handshaking principle states that in any graph, the sum of all vertex valences is twice the total number of edges. Write a sentence of two to explain why each of the following is a consequence of the handshaking principle with *n* people.
    a. The sum of all valences is an even number.
    b. In any graph, the number of vertices with odd valence is even.
    c. If all vertices in a graph have the same valence, *r*, then the graph has $\frac{1}{2}nr$ edges.

12. Draw graphs satisfying each of the following specifications:
    a. 6 vertices: 2 of valence 3, 2 of valence 4 and 2 of valence 5.
    b. 6 vertices: 1 of valence 2, 3 of valence 4 and 2 of valence 5.
    c. 6 vertices: 1 of valence 3, 4 of valence 4 and 1 of valence 5.

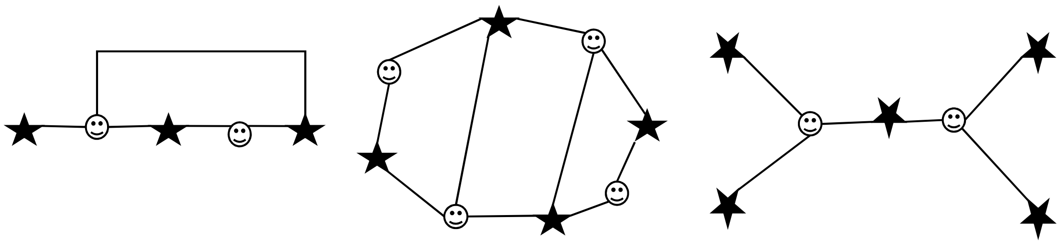
13.



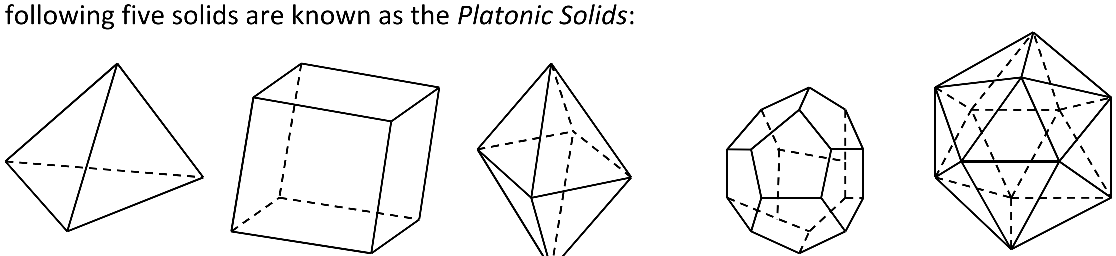
Of particular importance in coding theory are cube graphs, which may be constructed by taking as vertices all binary words (sequences of 0s and 1s) of a given length and joining two of these vertices if the corresponding binary words differ in just one place. The graph obtained in this way from the binary words of length *k* is called the ***k*-cube** (or *k-dimensional* cube), and is denoted $\boldsymbol{Q_k}$. Cube graphs for *k* = 1,2,3 and 4 are shown above. Give a formula for the number of edges of a $\boldsymbol{Q_k}$ graph.

14. A **bipartite graph** is a graph whose vertex-set can be split into sets *A* and *B* such that all edges in the graph join a vertex from *A* with a vertex from *B*. Under what condition will a bipartite graph have a

Hamiltonian circuit? How would you add vertices and edges to the graphs below (vetices are either stars or faces) to make bipartite graphs with Hamiltonian circuits?

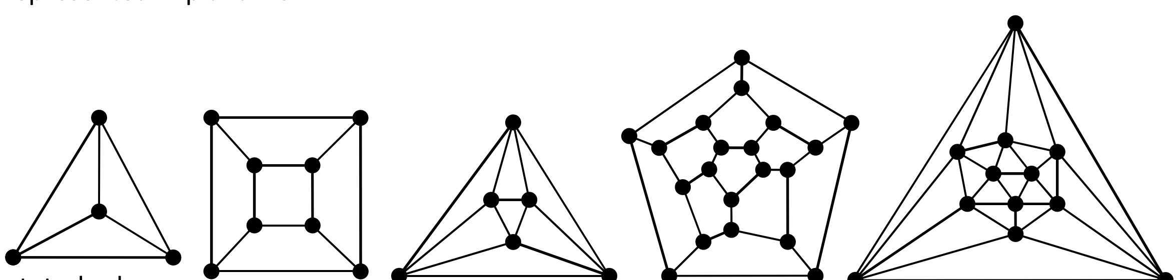


15. The following five solids are known as the *Platonic Solids*:



tetrahedron cube octahedron dodecahedron Icosahedron

If you consider the edges and vertices of these solids as the edges and vertices of a graph, each can be represented in planar form:



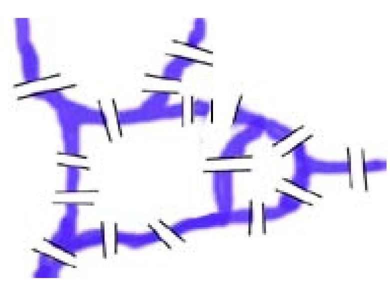
tetrahedron cube octahedron dodecahedron icosahedron

a. Which of the platonic graphs have an Euler circuit?

b. Which of the platonic graphs have a Hamiltonian circuit?

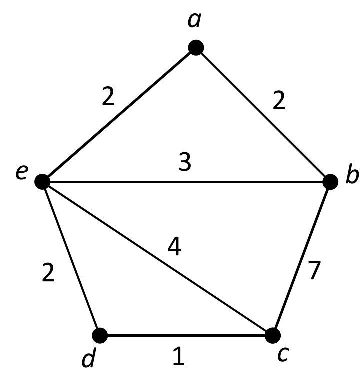
c. Euler's formula applies to polyhedra. It states that if *n*, *m* and *f* are the number of vertices, edges and faces, respectively, then $n - m + f = 2$. For example, for the tetrahedron, $4 - 6 + 4 = 2$. Verify Euler's formula for the other 4 platonic solids.

16. The town of Palmberg has rivers, bridges and islands as shown in the diagram.



a. Model the map as a graph where each separate land mass is represented by a vertex and each bridge is an edge.
b. Is there a there a way the people of Palmberg can walk along a path that will pass over each bridge exactly once? Explain.

17. A postman wishes to deliver mail along all the streets in his area and then return to his post office. The graph of the streets (edges) and intersections (vertices) is shown at right. How can the route be planned so as to minimize the total distance traveled?



18. How many nonisomorphic rooted trees are there with 5 vertices?

19. Consider the divisibility poset {1, 2, 5, 20, 50, 100}, | .
   a. Draw the Hasse diagram of this poset.
   b. Determine whether this poset is a lattice.

20. Prove or disprove: It is possible to inscribe a circle in any convex quadrilateral in which the sums of the opposite sides are equal.

Math 15 - Chapters 1 and 2 Test Solutions

1. Consider the declarative statements,
   P: The temperature outside is over $50°C$
   Q: John will fry an egg
   R: Mary is hungry
   Using logical connectives, write a proposition which symbolizes the following:
   (a) If it is over $50°C$ and Mary is hungry, then John will fry an egg on the side walk.
   ANS: $(P \wedge R) \rightarrow Q$
   (b) John will fry an egg on the side walk only if the temperature outside is over $50°C$
   ANS: This is equivalent to saying, "If John is frying an egg on the sidewalk, then it must be over $50°C$ outside. That is, Q is necessary for P or, in symbols, $Q \rightarrow P$.
   (c) The temperature outside is not over $50°C$.
   ANS: $\neg P$
   (d) If John does not fry an egg on the side walk, then either Mary is not hungry or it is not over $50°C$.
   ANS: $\neg Q \rightarrow (\neg R \vee \neg P)$

2. Determine which of the following statements is true.
   F_ Exactly one of these statements is false.
   F_ Exactly two of these statements are false.
   F_ Exactly three of these statements are false.
   F_ Exactly four of these statements are false.
   F_ Exactly five of these statements are false.
   F_ Exactly six of these statements are false.
   F_ Exactly seven of these statements are false.
   F_ Exactly eight of these statements are false.
   T_ Exactly nine of these statements are false.
   F_ Exactly ten of these statements are false.

3. Construct a truth table for the proposition $P = [p \rightarrow (q \vee r)] \wedge [\neg(p \leftrightarrow \neg r)]$.

p	q	r	$q \vee r$	$p \rightarrow (q \vee r)$	$\neg r$	$p \leftrightarrow \neg r$	$\neg(p \leftrightarrow \neg r)$	P
0	0	0	0	1	1	0	1	1
0	0	1	1	1	0	1	0	0
0	1	0	1	1	1	0	1	1
0	1	1	1	1	0	1	0	0
1	0	0	0	0	1	1	0	0
1	0	1	1	1	0	0	1	1
1	1	0	1	1	1	1	0	0
1	1	1	1	1	0	0	1	1

4. Is $\neg R \wedge (T \leftrightarrow P \leftrightarrow T \vee L)P$ a well formed expression? If not, how could you introduce new symbols into the express to make it a well-formed expression?
   ANS: Clearly were missing a logic connective between the parenthetical expression and the declarative, P. Notice that the expression preceding the final P is well formed, assuming that we are following the usual precedence of logical operators, so that the "or" operator has precedence over that equivalence operator, which is parsed from left to right. Adding parentheses to the expression as follows doesn't change its logic according to these precedence rules, but it does lend clarity to the expression: $\neg R \wedge \big((T \leftrightarrow P) \leftrightarrow (T \vee L)\big)$. Now any logical connective could be inserted between this expression and the subsequent declarative P. To be sure, what follows is the truth table for the expression was produced by the applet at http://logik.phl.univie.ac.at/~chris/cgi-bin/cgi-form?key=00000076 :

```
  L P Q R T |  ~R & ((P <-> Q) <-> (T v L))
-----------+-----------------------------
1 1 1 1 1  |  0 *0       1      1     1
1 1 1 1 0  |  0 *0       1      1     1
1 1 1 0 1  |  1 *1       1      1     1
1 1 1 0 0  |  1 *1       1      1     1
1 1 0 1 1  |  0 *0       0      0     1
1 1 0 1 0  |  0 *0       0      0     1
1 1 0 0 1  |  1 *0       0      0     1
1 1 0 0 0  |  1 *0       0      0     1
1 0 1 1 1  |  0 *0       0      0     1
1 0 1 1 0  |  0 *0       0      0     1
1 0 1 0 1  |  1 *0       0      0     1
1 0 1 0 0  |  1 *0       0      0     1
1 0 0 1 1  |  0 *0       1      1     1
1 0 0 1 0  |  0 *0       1      1     1
1 0 0 0 1  |  1 *1       1      1     1
1 0 0 0 0  |  1 *1       1      1     1
0 1 1 1 1  |  0 *0       1      1     1
0 1 1 1 0  |  0 *0       1      0     0
0 1 1 0 1  |  1 *1       1      1     1
0 1 1 0 0  |  1 *0       1      0     0
0 1 0 1 1  |  0 *0       0      0     1
0 1 0 1 0  |  0 *0       0      1     0
0 1 0 0 1  |  1 *0       0      0     1
0 1 0 0 0  |  1 *1       0      1     0
0 0 1 1 1  |  0 *0       0      0     1
0 0 1 1 0  |  0 *0       0      1     0
0 0 1 0 1  |  1 *0       0      0     1
0 0 1 0 0  |  1 *1       0      1     0
0 0 0 1 1  |  0 *0       1      1     1
0 0 0 1 0  |  0 *0       1      0     0
0 0 0 0 1  |  1 *1       1      1     1
0 0 0 0 0  |  1 *0       1      0     0
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5. Suppose we know A is true, $A \to (B \to C)$ is true and that $B \to D$ is true. Can we conclude that $\neg C \to D$? Explain…or, that is, prove your claim.
ANS: We are assuming that A is true, $A \to (B \to C)$ is equivalent to, and can be simplified as $B \to C$. Thus we can simplify our assumptions to $B \to C$ and $B \to D$. Using the implication rule, we can rewrite these as $\neg B \vee C$ and $\neg B \vee D$. Now, the implication rule also says that $\neg C \to D$ is equivalent to $C \vee D$.
See the truth table:

$\neg B$	C	D	$\neg B \vee C$	$\neg B \vee D$	$C \vee D$
0	1	1	1	1	1
1	0	0	1	1	0
1	1	1	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1

This shows that the conclusion does not follow, since in the case where *B*, *C*, and *D* are all false, $B \to C$ and $B \to D$ are both true, but $\neg C \to D$ is false. We could also do a deductive proof:

	statement	Reason
1	A	Given
2	$A \to (B \to C)$	Given
3	$B \to D$	Given
4	$B \to C$	Modus Ponens (2) (3)
5	$B \to (C \wedge D)$	Addition (3) and (4)
6	$\neg B \vee (C \wedge D)$	Implication (5)
7	$(\neg B \vee C) \wedge (\neg B \vee D)$	Distributive property
8	$\neg C \to \neg B$ and $\neg D \to \neg B$	Contrapositive (3) and (4)
9	Uh…yeah…	…just cuz.

6. Given that $\neg A \rightarrow B$ and $B \rightarrow A$ and $A \rightarrow \neg B$ can we conclude $\wedge\neg B$? Use the method of indirect proof to prove or disprove.
   ANS:

	statement	Reason
1	$\neg A \rightarrow B$	Given
2	$B \rightarrow A$	Given
3	$A \rightarrow \neg B$	Given
4	$\neg A \rightarrow \neg B$	Contrapositive (2)
5	$\neg(\neg A)$	Absurdity (1) and (4)
6	A	Double negative (5)
7	$B \rightarrow \neg A$	Contrapositive and DN (3)
8	$\neg B$	Absurdity (2) and (7)
9	$A \wedge \neg B$	Addition (6) and (8)

7. Prove that implication is not associative.
   ANS: That is, we want to show that $(A \rightarrow B) \rightarrow C$ is not equivalent to $A \rightarrow (B \rightarrow C)$.
   The implication rule tells us that $(A \rightarrow B) \rightarrow C$ is equivalent to $\neg(\neg A \vee B) \vee C \leftrightarrow$
   which is equivalent to $(A \wedge \neg B) \vee C$ by de Morgan's laws and distributing the or, we have the equivalent statement $(A \vee C) \wedge (\neg B \vee C)$.
   On the other hand, $A \rightarrow (B \rightarrow C)$ is equivalent to $\neg A \vee (\neg B \vee C) = \neg A \vee \neg B \vee C$. To be sure, the last two columns of this truth table show these are not equivalent:

A	B	C	$A \vee C$	$\neg B \vee C$	$(A \vee C) \wedge (\neg B \vee C)$	$\neg A \vee \neg B \vee C$
0	0	0	0	1	0	1
0	0	1	1	1	1	1
0	1	0	0	0	0	1
0	1	1	1	1	1	1
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

8. What are the axioms of the propositional logic?
   ANS: These are the basic building blocks of propositional logic that are assumed to be true without proof.

   The Hunter text covers propositional logic and predicate logic before discussing axiomatic systems. In fact, the book doesn't address this question directly. So we need a set and actions on the set that we assume to be self-evident, requiring no proof. Let the set be all declarative statements without qualifiers. Then the derivation rules laid out on pages 18 and 19 in Hunter can serve as the axioms. Double negation, implication, De Morgan's Laws, commutivity, associativity, conjunction, modus ponens, mudus tollens, simplification and addition. These are given without proof and all proof sequences are based on these, so these are the axioms of propositional logic.

   Of course, different practitioners may vary these axioms and typically strive to keep the number of axioms to a minimum, so if one axiom can be deduced (proved) using the other axioms, it is removed from the set of axioms.

9. Prove or disprove: $(A \to B) \vee (A \to C) \to B \vee C$ is equivalent to $A \vee B \vee C$.

ANS: The knucklehead approach is to do the truth table and see that their logic is the same.

A	B	C	$A \to B$	$A \to C$	$(A \to B) \vee (A \to C)$	$B \vee C$	$(A \to B) \vee (A \to C) \to B \vee C$
0	0	0	1	1	1	0	0
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	1
1	0	1	0	1	1	1	1
1	1	0	1	0	1	1	1
1	1	1	1	1	1	1	1

Yep…it's actually harder to work it out with the rules, but let's give it a go:

	statement	Reason
1	$(A \to B) \vee (A \to C) \to B \vee C$	Given
2	$\neg\big((\neg A \vee B) \vee (\neg A \vee C)\big) \vee (B \vee C)$	Implication
3	$\big(\neg(\neg A \vee B) \wedge \neg(\neg A \vee C)\big) \vee (B \vee C)$	De Morgan
4	$(A \wedge \neg B \wedge A \wedge \neg C) \vee (B \vee C)$	De Morgan
5	$(A \wedge \neg B \wedge \neg C) \vee (B \vee C)$	Simplification
6	$A \vee (B \vee C) \wedge \neg B \vee (B \vee C) \wedge \neg C \vee (B \vee C)$	Distributive
7	$(A \vee B \vee C) \wedge (\neg B \vee B \vee C) \wedge (\neg C \vee B \vee C)$	Associative
8	$A \vee B \vee C$	Simplification

10. First, translate the following argument into propositional logic. Then, prove that the argument is valid using the method of formal derivation. Explain your answer.

Sally is destined to be either a fearless adventurer or a great psychiatrist. If Robert is not paranoid then Sally is not destined to be a great psychiatrist. Yet Sally is clearly not destined to be fearless adventurer. So Robert is definitely paranoid.

ANS: Let P: *Sally is destined to be a fearless adventurer*

Let Q: *Sally is destined to be a great psychologist*

Let R: *Robert is paranoid* Given $P \vee Q$, $\neg R \to \neg Q$, and $\neg P$ prove R

	statement	Reason
1	$P \vee Q$	Given
2	$\neg R \to \neg Q$	Given
3	$\neg P$	Given
4	$Q \to R$	Contrapositive (2)
5	$\neg P \to Q$	Implication
6	Q	Modus ponens (3) (5)
7	R	Modus ponens (4) (6)

11. The handshaking principle states that in any graph, the sum of all vertex valences is twice the total number of edges. Write a sentence of two to explain why each of the following is a consequence of the handshaking principle with *n* people.
    a. The sum of all valences is an even number.
    ANS: The total number of edges is a natural (counting) number and twice any natural number is an even number.
    b. In any graph, the number of vertices with odd valence is even.
    ANS: ANS: Since every even number has a factor of two we can use the distributive property to factor two from each term in the sum of all even numbers and thus the sum of even number is also even. For example, 2+18+28+22 = 2(1 + 9 + 14 + 11) = 2(35) = 70 is even. The sum of two

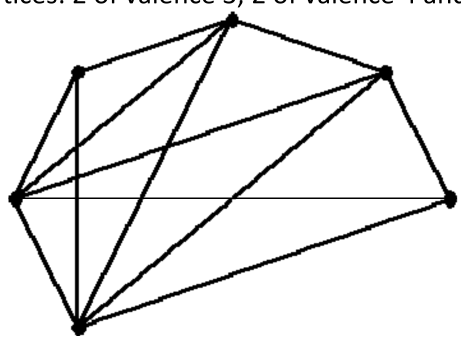
odd numbers is even so the sum of an even number of odd numbers is even (you can pair them up so the sum reduces to a sum of even numbers.) Thus, the sum of an odd number of odd numbers is an odd number plus the sum of an even number of odd numbers, or the sum of an odd and an even, which is odd. But we know that the sum of all valences is even, so there cannot be an odd number of odd valences.

c. If all vertices in a graph have the same valence, *r*, then the graph has $\frac{1}{2}nr$ edges.

On a graph with n vertices each with valence r, the fundamental principle of counting tells us there will be a total of nr valences. Since there are 2 valences per edge, there will be $nr/2$ edges.

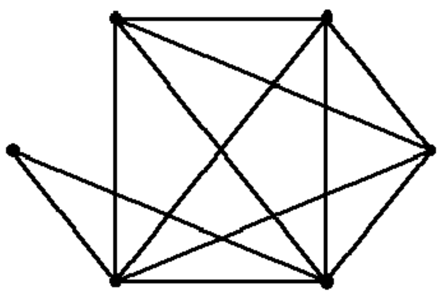
12. Draw graphs satisfying each of the following specifications:

a. 6 vertices: 2 of valence 3, 2 of valence 4 and 2 of valence 5.



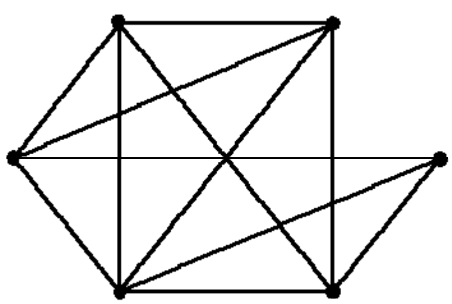
ANS:

b. 6 vertices: 1 of valence 2, 3 of valence 4 and 2 of valence 5.



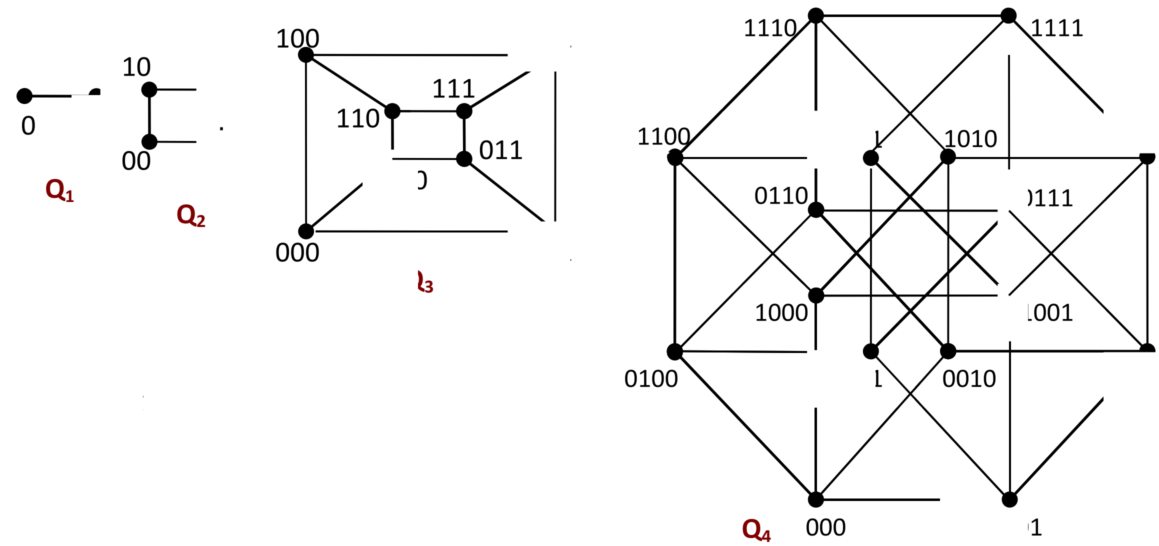
ANS:

c. 6 vertices: 1 of valence 3, 4 of valence 4 and 1 of valence 5.



ANS:

13.

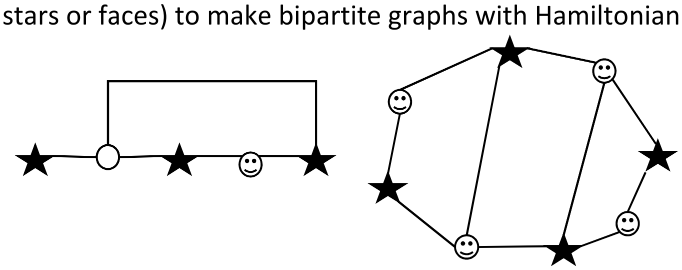


Of particular importance in coding theory are cube graphs, which may be constructed by taking as vertices all binary words (sequences of 0s and 1s) of a given length and joining two of these vertices if the corresponding binary words differ in just one place. The graph obtained in this way from the binary words of length k is called the ***k*-cube** (or *k-dimensional* cube), and is denoted $\boldsymbol{Q_k}$. Cube graphs for k = 1,2,3 and 4 are shown above. Give a formula for the number of edges of a $\boldsymbol{Q_k}$ graph.

ANS: The number of vertices of can be computed using the fundamental principle of counting or the formula for handshakes: The number of vertices is the number of k-digit binary numbers or . Since, by definition, the valence of each vertex is the number of digits, k, we have

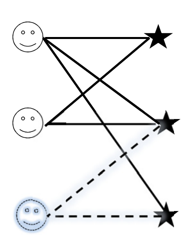
edges.

14. A **bipartite graph** is a graph whose vertex-set can be split into sets A and B such that all edges in the graph join a vertex from A with a vertex from B. Under what condition will a bipartite graph have a Hamiltonian circuit? How would you add vertices and edges to the graphs below (vertices are either stars or faces) to make bipartite graphs with Hamiltonian circuits?



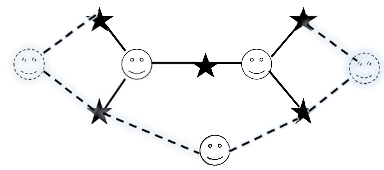
ANS: A necessary condition for a bipartite graph to have a Hamiltonian circuit is that there is the same number of vertices in each group (i.e. as many stars as there are faces.)

The first one requires the addition of one face and two edges (shown here as the dotted parts).

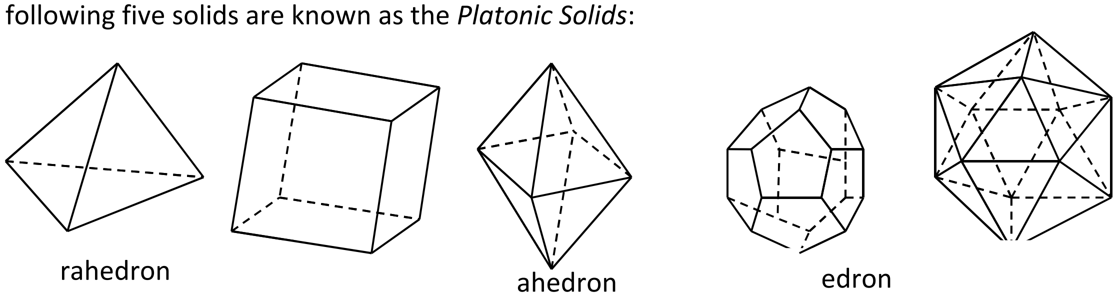


The second one requires no new vertices or edges: A Hamiltonian path is found by simply traversing the outer circumference

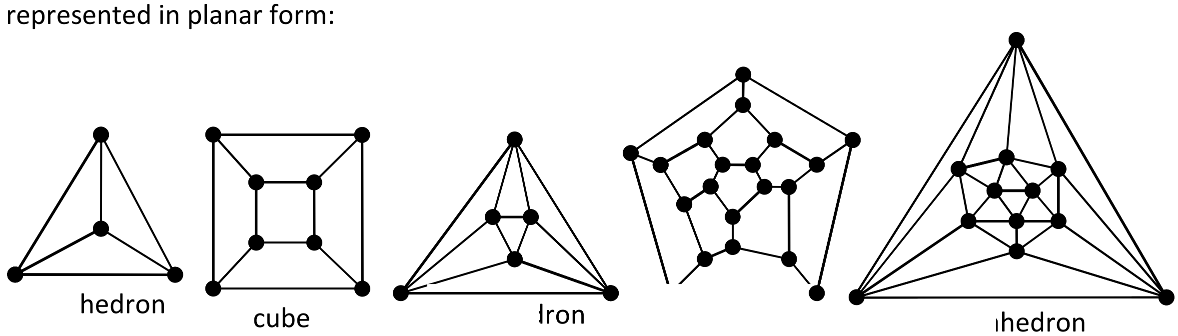
The third one requires three new face vertices (faces) and six new edges (shown as dashed lines.)

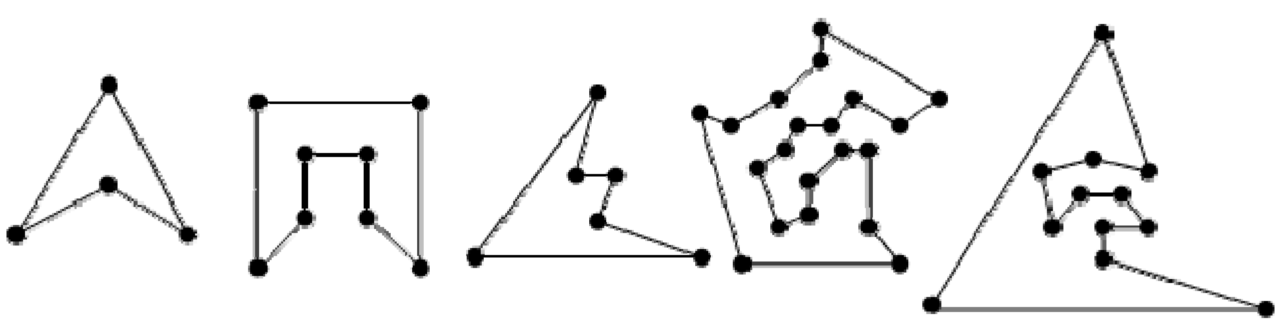


15. The following five solids are known as the *Platonic Solids*:



If you consider the edges and vertices of these solids as the edges and vertices of a graph, each can be represented in planar form:





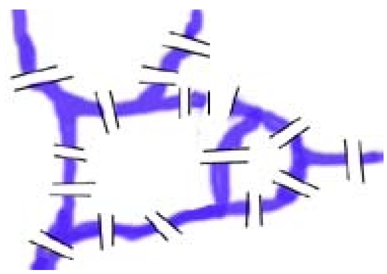
a. Which of the platonic graphs have an Euler circuit?
   ANS: Only the octahedron has all even valences, thus it is the only one with an Euler circuit.

b. Which of the platonic graphs have a Hamiltonian circuit?
   ANS: They all have Hamiltonian circuits, as demonstrated by the subgraphs above.

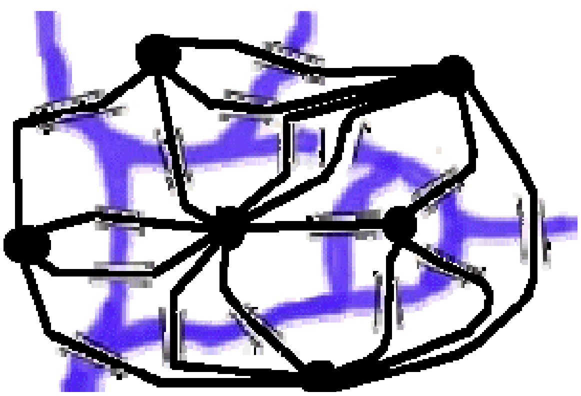
c. Euler's formula applies to polyhedra. It states that if *n*, *m* and *f* are the number of vertices, edges and faces, respectively, then $n - m + f = 2$. For example, for the tetrahedron, $4 - 6 + 4 = 2$. Verify Euler's formula for the other 4 platonic solids. ANS: We can tabulate the results as follows:

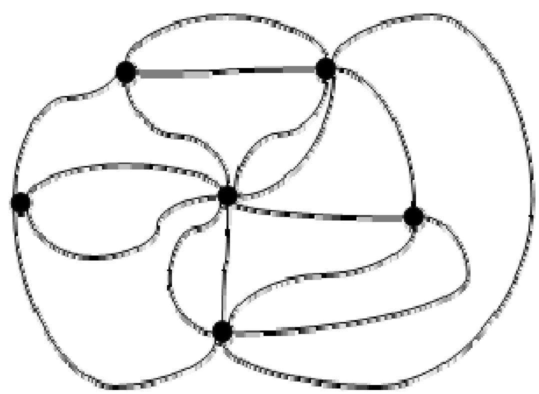
	Face	Vertices	Edges	Formula
Tetrahedron	4 triangular	4	6	4+4-6=2
Hexahedron (cube)	6 square	8	12	6+8-12=2
Octahedron	8 triangular	6	12	8+6-12=2
Dodecahedron	12 pentagonal	20	30	12+20-30=2
Icosahedron	20 triangular	12	30	20+12-30=2

16. The town of Palmberg has rivers, bridges and islands as shown in the diagram.
    a. Model the map as a graph where each separate land mass is represented by a vertex and each bridge is an edge.
    b. Is there a there a way the people of Palmberg can walk along a path that will pass over each bridge exactly once? Explain.

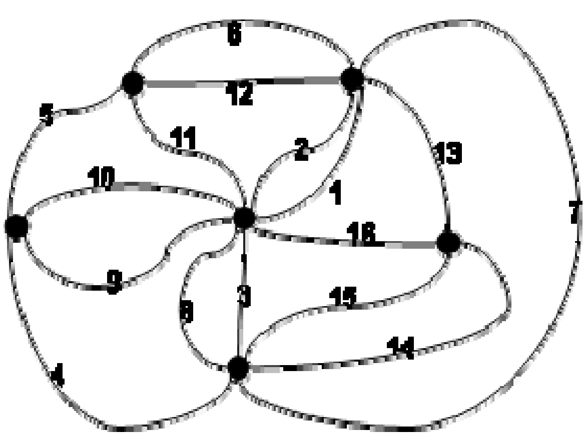


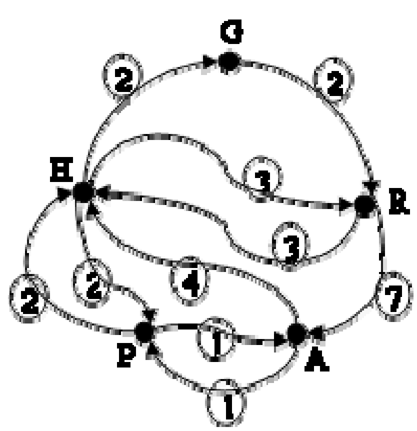
ANS: It may help to first draw the graph on the map and then note the vertices and there valences to reproduce the graph:



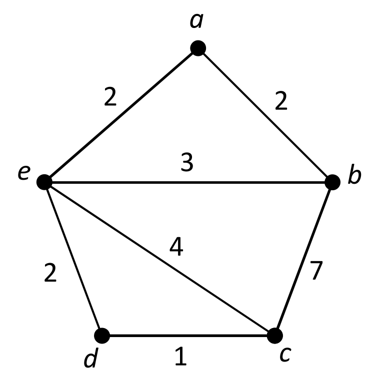


Since all the valences are even, Euler's theorem guarantees an Euler circuit. To be sure, here's one:

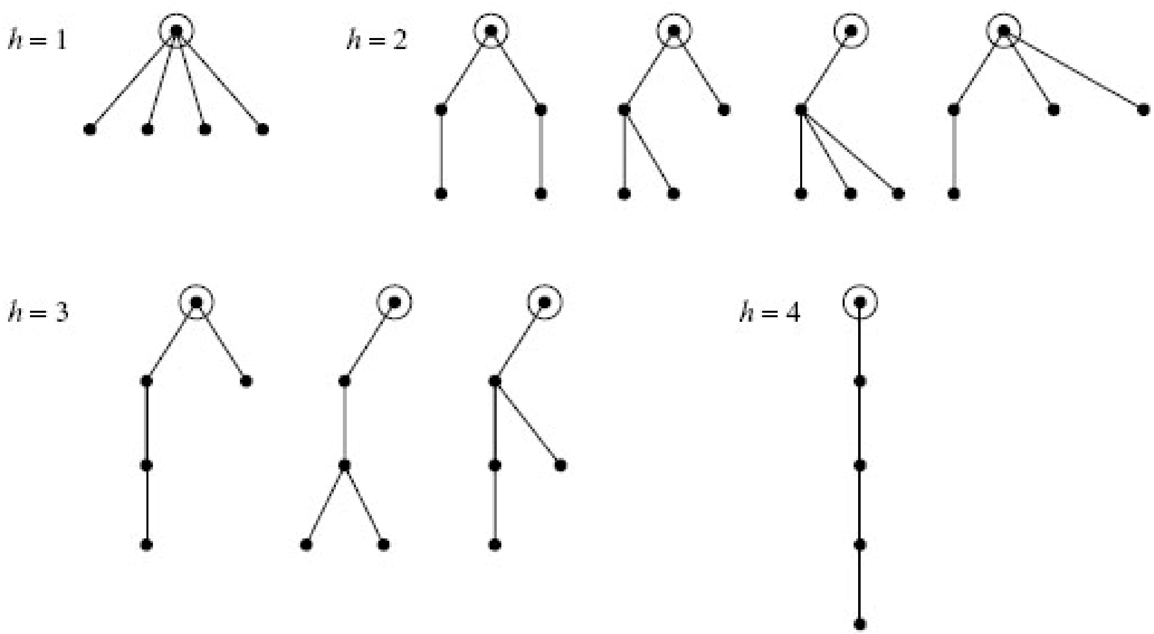




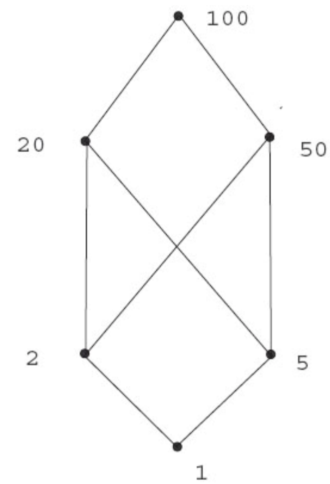
17. A postman wishes to deliver mail along all the streets in his area and then return to his post office. The graph of the streets (edges) and intersections (vertices) is shown at right. How can the route be planned so as to minimize the total distance traveled?
ANS: As is, the total edge length of the graph is 21. We want to Eulerize the graph in such a way that we minimize the increase in total edge length. We could Eulerize by duplicating edge bc with length 7, but duplicating edges cd, de and eb also eulerizes the graph and only adds 6 to the total edge length. The Eulerized graph is shown at right above. The Euler circuit could be GRAPHRHPAHG which has length 27.



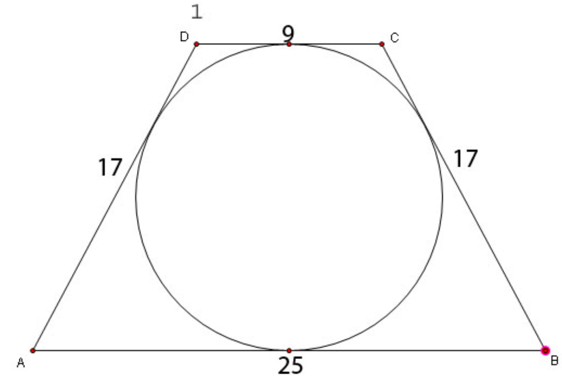
18. How many non-isomorphic rooted trees are there with 5 vertices?
ANS: These can be sorted by their heights, as shown. There is one tree of height 1, 4 trees of height 2, 3 trees of height 3 and 1 tree of height 4, for a total of 9 non-isomorphic trees:



19. Consider the divisibility poset {1, 2, 5, 20, 50, 100}, | .
    a. Draw the Hasse diagram of this poset.
       ANS: See diagram at right.
    b. Determine whether this poset is a lattice.
       ANS: Recall that a poset is a lattice if every pair of elements $x, y \in X$ has both a meet and a join. It is not a lattice. For example, the meet of 20 and 50, denoted $20 \wedge 50$ does not exist. It's not 1 since 1|20, 1|50 but so does 5|20, 5|50 and 2|20, 2|50, so there is no greatest lower bound $glb\{20,50\}$.



20. Prove or disprove: It is possible to inscribe a circle in any convex quadrilateral in which the sums of the opposite sides are equal.
    ANS: Consider a related problem: find the altitude of an isosceles trapezoid with bases 9 and 25 cm. and one side of length 17 cm. To solve the problem, a student inscribes a circle in the trapezoid stating that this was possible by virtue of the theorem that in any quadrilateral circumscribed about a circle the sums of the opposite sides



are equal, which is true in the given trapezoid (9 + 25 = 17 + 17). The student then determined the altitude as the diameter of a the circle inscribed in the isosceles trapezoid, which–as had been proved in a problem solved earlier—is the mean proportional between the two bases $= \sqrt{9 \cdot 25} = 15$.

But there is a flaw in the reasoning here.Correct reasoning is required. The correctness of a proof depends on the truth or falsity of the deductions that in entails. Consider the following deduction, for example:

1) All rectangles have two equal diagonals.
2) All squares are rectangles.
3) Conclusion: All squares have equal diagonals.A more abstract way of saying similar things is characterized as

1) All *M* are *P*.
2) All *S* are *M*.
3) Conclusion: All *S* are *P*.

In set notation we'd say that $M \subset P$ and $S \subset M$ allows us to conclude that $S \subset P$.

Here's a similar deduction with a negative conclusion:

1) No quadrilateral in which the sum of opposite angles is different from 180 cab be inscribed in a circle.
2) In an oblique parallelogram the sum of the opposite angles is not equal to 180.
3) Conclusion: No oblique parallelogram can be inscribed in a circle.

Many deductions in geometry follow one of these two patterns.

Now let's look at an incorrect reasoning.

1) In all quadrilaterals circumscribed about a circle the sums of the opposite sides are equal.
2) In a given trapezoid the sums of the opposite sides are equal.
3) Conclusion: The give trapezoid can be circumscribed about a circle.

Let *P* be the set of all quadrilaterals that can be circumscribed about a circle.
Let *M* be the set of all quadrilaterals in which the sum of opposite sides are equal.
Let *S* be the set of all trapezoids in which the sum of bases is equal to the sum of lateral sides.

The incorrect reasoning above could be written more succinctly as

1) All *P* are *M*.
2) All *S* are *M*.
3) Conclusion: All *S* are *P*.

The error above stems from confusing a direct theorem with its converse.

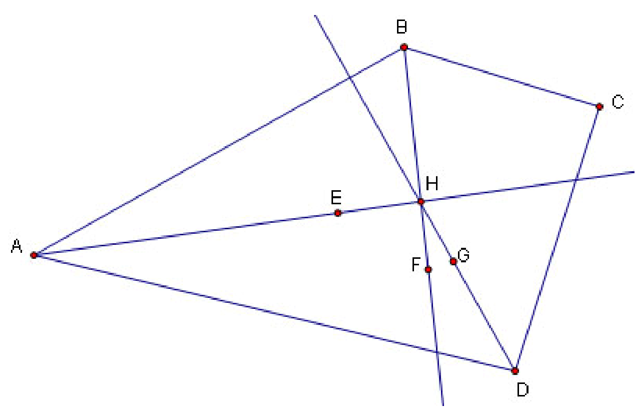
The direct theorem states that

"In any quadrilateral in which a circle can be inscribed, the sums of opposite sides are equal."

This must be carefully distinguished from the converse,

"A circle can be inscribed in any quadrilateral in which the sums of opposite sides are equal."

To be sure, neither statement is necessarily the direct one – either one of the two is the converse of the other. Let's prove the last statement (the converse):



Proof: If a circle can be inscribed in a quadrilateral, then its center will be equidistant from all four sides. Since the set of all points equidistant from the sides of an angle form the bisector of the angle, the center of inscribed circle must lie on the bisector of each interior angle. So the center of the inscribe circle is the point of intersection of the four bisectors of the interior angles of the quadrilateral. Suppose that the bisectors of three interior angles are concurrent, as in the figure at right. That is, *AH* bisects $\angle BAD$, *DH* bisects $\angle ADC$, and *BH* bisects $\angle ABC$. Then it can be shown that (prove this) the fourth angle bisector Must also be concurrent with those (in the figure above, that means that the angle bisector at *C* will pass through *H*.

The given information about quadrilateral *ABCD* here is that $AD + BC = AB + CD$. If the quadrilateral is a rhombus, then the diagonals are perpendicular bisectors and their intersection is at once seen to be the center of the inscribed circle; so we can always inscribe a circle in a rhombus. So assume *ABCD* is not a rhombus so there are two adjacent unequal sides, say $AB > BC$. Then since $AD + BC = AB + CD$ it follows that $CD < AD$. On AB mark off $BJ = BC$ and draw JC to produce isosceles triangle ΔBCJ. Similarly mark off $DK = DC$ on AD to produce isosceles triangle ΔDCK. We now show that ΔAJK is isosceles. Since $AD + BC = AB + CD \leftrightarrow AD - CD = AB - BC$. But $AD - CD = AK$ and $AB - BJ = AJ$, so $AK = AJ$, making ΔAJK isosceles. This means that $AH \perp JK$, $BH \perp CJ$ and $DH \perp CK$. So H is the center of the inscribed circle.