Math 15 – Discrete Structures – §3.2 – Homework 9 Solutions

3.2#20: Is $1+\frac{17}{6}n-2n^2+\frac{7}{6}n^3$ a closed form solution to the recurrence relation,

$$P(n)=\begin{cases}1 & \text{if } n=0\\ 4\cdot P(n-1)-n^2 & \text{if } n>0\end{cases}$$

ANS: Substituting, $4\cdot P(n-1)-n^2=4\left(1+\frac{17}{6}(n-1)-2(n-1)^2+\frac{7}{6}(n-1)^3\right)-n^2$

$$=4\left(1+\frac{17}{6}n-\frac{17}{6}-2(n^2-2n+1)+\frac{7}{6}(n^3-3n^2+3n-1)\right)-n^2$$

$$=4\left(1-\frac{17}{6}-2-\frac{7}{6}+\frac{17}{6}n+4n+\frac{7}{2}n-2n^2-\frac{7n^2}{2}+\frac{7n^3}{6}\right)-n^2$$

$$=4\left(\frac{-30}{6}+\frac{31}{3}n-\frac{11}{2}n^2+\frac{7n^3}{6}\right)-n^2$$

$$=-20+\frac{124}{3}n-23n^2+\frac{14n^3}{3}\neq P(n)$$

To be sure, entering the following in Mathematica produces an equivalent result:

$f[\text{n\_}]:=1+17*n/6-2*n^2+7*n^3/6$

$4*f[n-1]-n^2$

$$4\left(\frac{7}{6}(n-1)^3-2(n-1)^2+\frac{17(n-1)}{6}+1\right)-n^2$$

Simplify[%]

$$\frac{1}{3}(14n^3-69n^2+124n-60)$$

Interestingly, the difference between this polynomial and $P(n)$ is 0 at n = 0,1, 2, 3, but this breaks down at 4:

n	$f(n)$	$P(n)$
0	1	1
1	3	3
2	8	8
3	23	23
4	55	76

Out of curiosity, try the following in Mathematica:

RSolve[p[n]-4p[n-1]+n^2==0,p[n], n]

$\{\{p(n)\to c_1\, 4^{n-1}+4/27\,((9\,n^2)/4+6\,n-9\,4^n-5\,4^{n+1}+3\,2^{2\,n+3}+5)\}\}$

Imposing the initial condition, p(0)=1 we have

$$P(0)=\frac{c_1}{4}-\frac{4}{27}(9-20+24+5)=\frac{c_1}{4}-\frac{8}{3}=1\Leftrightarrow c_1=\frac{44}{3}$$

Thus $P(n)=\frac{11}{3}\cdot 4^n+\frac{4}{27}\left(\frac{9}{4}n^2+6n-9\cdot 4^n-20\cdot 4^n+24\cdot 4^n+5\right)=\frac{7}{27}\cdot 4^n+\frac{1}{3}n^2+\frac{8}{9}n+\frac{20}{27}$

That works! How does Mathematica find this closed form??

3.2#22: Let $f(n) = An^2 + Bn + C$.

Then $f(n+1) - f(n) = A(n+1)^2 + B(n+1) + C - (An^2 + Bn + C) = 2An + A + B$ is linear in n.

3.2#23: Recall the definition of Fibonacci numbers: $F(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = 2 \\ F(n-1) + F(n-2) & \text{if } n > 2 \end{cases}$

(a) Compute the sequence of differences of the first nine Fibonacci numbers. What seems to be true about this sequence?

ANS:

n	1	2	3	4	5	6	7	8	9
$F(n+1) - F(n)$	0	1	1	2	3	5	8	13	21

…wow, it seems like the differences are the sequence again!

(b) Proof: The difference of successive terms is the term preceding, so it repeats the sequence:

$$F(n+1) - F(n) = \big(F(n) + F(n-1)\big) - F(n) = F(n-1)$$

(c) Explain why a closed-form solution for the Fibonacci numbers cannot be a polynomial function.

ANS: We've seen that if the closed form solution of recursive function has a periodic difference if and only if the closed form solution is polynomial. In this case repeating the difference will just lead to another Fibonacci sequence. Thus no degree of difference will lead to a constant, so the closed form solution cannot be a polynomial.

3.2#24: Suppose you are given a sequence of numbers, $a_1, a_2, \dots, a_k$. Explain how to construct a polynomial $p(x)$ such that $p(n) = a_n$, for all $n = 1,2,\dots,k$. Note that this fact, along with Exercise #23, shows that it is possible for a closed form formula to match a recurrence relation for arbitrarily many terms, without being a valid closed-form solution. The system of k equations in k unknowns, $\{f(i) = a_i\}_{i=1}^{k}$, where $f(x)$ is an arbitrary polynomial of degree $k-1$. For example, given 3,5,8,13, we solve the system

$$\begin{aligned} c_3 + c_2 + c_1 + c_0 &= 3 \\ 8c_3 + 4c_2 + 2c_1 + c_0 &= 5 \\ 27c_3 + 9c_2 + 3c_1 + c_0 &= 8 \\ 64c_3 + 16c_2 + 4c_1 + c_0 &= 13 \end{aligned}$$

This is equivalent to the matrix equation,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 8 \\ 13 \end{pmatrix}$$

This matrix equation can be solved by left-multiplying both sides by the inverse of the coefficient matrix. On the left side this produces the identity matrix so we have

$$\begin{pmatrix} c_3 \\ c_2 \\ c_1 \\ c_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 5 \\ 8 \\ 13 \end{pmatrix}$$

Let's use Reduce (http://reduce-algebra.sourceforge.net/) to solve the system. The screen captures below illustrate how this could be done using Reduce.

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36: (mat((1,1,1,1),(8,4,2,1),(27,9,3,1),(64,16,4,1)))^(-1);
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$$\begin{pmatrix} \frac{-1}{6} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{6} \\ \frac{3}{2} & -4 & \frac{7}{2} & -1 \\ \frac{-13}{3} & \frac{19}{2} & -7 & \frac{11}{6} \\ 4 & -6 & 4 & -1 \end{pmatrix}$$

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39: (mat((1,1,1,1),(8,4,2,1),(27,9,3,1),(64,16,4,1)))^(-1)*(mat((3),(5),(8),(13)
));
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$$\begin{pmatrix} \frac{1}{6} \\ \frac{-1}{2} \\ \frac{7}{3} \\ 1 \end{pmatrix}$$

So $P(n) = \frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{7}{3}n + 1$ will do the job. To be sure, check that $P(1) = \frac{1-3+14+6}{6} = 3$, and so on.