3.1#6: Consider the recurrence relation \( P(n) = \begin{cases} 0 & \text{if } n = 0 \\ ([P(n-1)]^2 - n) & \text{if } n > 0 \end{cases} \)

Use the recurrence relation to compute

\[
\begin{align*}
P(1) &= (P(0))^2 - 1 = 0 - 1 = -1 \\
P(2) &= (P(1))^2 - 2 = 1 - 2 = -1 \\
P(3) &= (P(2))^2 - 3 = 1 - 3 = -2 \\
P(4) &= (P(3))^2 - 4 = 4 - 4 = 0
\end{align*}
\]

This raises the obvious question: Is there a closed form formula? Maybe, maybe not…

3.1#13: Circles can be packed into the shape of an equilateral triangle. Let \( T(n) \) be the number of circles needed to form a triangle with \( n \) circles on each edge. Experimenting with coins we see that \( T(2) = 3 \) and \( T(3) = 6 \). Write a recurrence relation for \( T(n) \).

ANS: \( T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n-1) + n & \text{if } n > 1 \end{cases} \)

These are the triangle numbers: \( 1 + 2 + 3 + \cdots + n \)

3.1#14: Let \( H(n) = \begin{cases} 1 & \text{if } n = 1 \\ \frac{1}{2}(H(n-1) + 6n - 6) & \text{if } n > 1 \end{cases} \)

and \( T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n-1) + n & \text{if } n > 1 \end{cases} \). Write \( H(n) \) in terms of \( T(n-1) \) and explain your reasoning.

ANS: \( H(n) \) = the number of circles needed form a hexagon with \( n \) circles at each edge. Each hexagon can be seen as 6 overlapping triangles with \( n \) circles on each edge. Each triangle has \( T(n) \) circles, so

\[
H(n) = 6 \cdot T(n) - 6n + 1 = 6(T(n-1) + n) - 6n + 1 = 6T(n-1) + 1
\]

This accounts for 6 triangles overlapping on an edge containing \( n \) circle and then adding back in the center circle which is on each of the overlapping.

Alternatively, think of 6 copies of \( T(n-1) \) fit around a single center circle. This situation is illustrated at right for \( n = 3 \). You see six groups of \( T(2) = 3 \) circles grouped around a center circle.

3.1#20: Give a recurrence relation that describes the sequence 3, 6, 12, 24, 48, 96, 192, …

ANS: \( S(n) = \begin{cases} 3, \text{ if } n = 1 \\ 2S(n-1), \text{ if } n > 1 \end{cases} \)

3.1#22: Let \( f: N \to R \) be any function on the natural numbers. The sum of the first \( n \) values of \( f(n) \) is written in sigma notation as \( \sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n) \).

(a) Write \( 1^2 + 2^2 + \cdots + n^2 \) in sigma notation: \( \sum_{k=1}^{n} k^2 \)

ANS: \( \sum_{k=1}^{n} k^2 \)

(b) Give a recurrence relation for \( \sum_{k=1}^{n} f(k) \) \( \text{ANS: } S(n) = \begin{cases} f(1), \text{ if } n = 1 \\ S(n-1) + f(n), \text{ if } n > 1 \end{cases} \)

3.1#25: Suppose we model the spread of a virus in a certain population as follows: on day 1, one person is infected. On each subsequent day, each infected person gives the cold to two others.

(a) Write down a recurrence relation for this model.

ANS: \( I(d) = \begin{cases} 1, \text{ if } d = 1 \\ 3 \cdot I(d-1), \text{ if } d > 1 \end{cases} \)

(b) What are some of the limitations of this model? How does it fail to be realistic?

ANS: No one ever recovers, the population is unlimited and no one ever dies.
Let $X$ be a set with $n$ elements. Let $E \subseteq \mathcal{P}(X)$ be the set of all subsets of $X$ with an even number of elements and $O \subseteq \mathcal{P}(X)$ be the subsets of $X$ with an odd number of elements. Let $E(n) = |E|$ and $O(n) = |O|$.

(a) Find a recurrence relation for $E(n)$ in terms of $O(n-1)$ and $E(n-1)$.

\[ E(n) = \begin{cases} 1 & \text{if } n = 1 \\ E(n-1) + O(n-1) & \text{if } n > 1 \end{cases} \]

(b) Find a recurrence relation for $O(n)$ in terms of $O(n-1)$ and $E(n-1)$.

\[ O(n) = \begin{cases} 1 & \text{if } n = 1 \\ O(n-1) + E(n-1) & \text{if } n > 1 \end{cases} \]

(c) Since $|\mathcal{P}(X)| = 2^n$ it is easy to see that $|E| = |O| = 2^{n-1}$. 

\[ E \subseteq \mathcal{P}(X) \]

\[ \mathcal{P}(X) \]