Math 15 – Discrete Structures – §3.1 – Homework 8 Solutions

3.1#6: Consider the recurrence relation $P(n) = \begin{cases} 0 & \text{if } \quad n = 0 \\ [P(n-1)]^2 - n & \text{if } \quad n > 0 \end{cases}$

Use the recurrence relation to compute

$$P(1) = \left(P(0)\right)^2 - 1 = 0 - 1 = -1$$
$$P(2) = \left(P(1)\right)^2 - 2 = 1 - 2 = -1$$
$$P(3) = \left(P(2)\right)^2 - 3 = 1 - 3 = -2$$
$$P(4) = \left(P(3)\right)^2 - 4 = 4 - 4 = 0$$

This raises the obvious question: Is there a closed form formula? Maybe, maybe not…

3.1#13: Circles can be packed into the shape of an equilateral triangle. Let $T(n)$ be the number of circles needed to form a triangle with *n* circles on each edge. Experimenting with coins we see that $T(2) = 3$ and $T(3) = 6$. Write a recurrence relation for $T(n)$.

ANS: $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n-1) + n & \text{if } n > 1 \end{cases}$ These are the triangle numbers: $1 + 2 + 3 + \cdots + n$

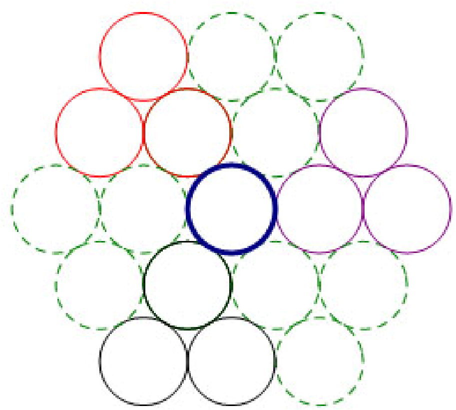
3.1#14: Let $H(n) = \begin{cases} 1 & \text{if } n = 1 \\ H(n-1) + 6n - 6 & \text{if } n > 1 \end{cases}$ and $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n-1) + n & \text{if } n > 1 \end{cases}$. Write $H(n)$ in terms of $T(n-1)$ and explain your reasoning.

ANS: $H(n) =$ the number of circles needed form a hexagon with *n* circles at each edge. Each hexagon can be seen as 6 overlapping triangles with *n* circles on each edge. Each triangle has $T(n)$ circles, so

$$H(n) = 6 \cdot T(n) - 6n + 1 = 6(T(n-1) + n) - 6n + 1$$
$$= 6T(n-1) + 1$$

This accounts for 6 triangles overlapping on an edge containing *n* circle and then adding back in the center circle which is on each of the overlapping.

Alternatively, think of 6 copies of $T(n-1)$ fit around a single center circle. This situation is illustrated at right for *n* = 3. You see six groups of *T*(2) = 3 circles grouped around a center circle.



3.1#20: Give a recurrence relation that describes the sequence 3, 6, 12, 24, 48, 96, 192, …

ANS: $S(n) = \begin{cases} 3, \ n = 1 \\ 2S(n-1), \ n > 1 \end{cases}$

3.1#22: Let $f: N \to R$ be any function on the natural numbers. The sum of the first *n* values of $f(n)$ is written in sigma notation as $\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n)$.

(a) Write $1^2 + 2^2 + \cdots + n^2$ in sigma notation: ANS: $\sum_{k=1}^{n} k^2$

(b) Give a recurrence relation for $\sum_{k=1}^{n} f(k)$ ANS: $S(n) = \begin{cases} f(1), \ n = 1 \\ S(n-1) + f(n), \ n > 1 \end{cases}$

3.1#25: Suppose we model the spread of a virus in a certain population as follows: on day 1, one person is infected. On each subsequent day, each infected person gives the cold to two others.

(a) Write down a recurrence relation for this model.

ANS: $I(d) = \begin{cases} 1 & \text{if } d = 1 \\ 3 \cdot I(d-1) & if\ d > 1 \end{cases}$

(b) What are some of the limitations of this model? How does it fail to be realistic?

ANS: No one ever recovers, the population is unlimited and no one ever dies.

3.1#26: Let X be a set with n elements. Let $E \subseteq \mathcal{P}(X)$ be the set of all subsets of X with an even number of elements and $\mathcal{O} \subseteq \mathcal{P}(X)$ be the subsets of X with an odd number of elements. Let $E(n) = |E|$ and $O(n) = |\mathcal{O}|$.

(a) Find a recurrence relation for $E(n)$ in terms of $O(n-1)$ and $E(n-1)$.

ANS: To get a feeling for this, look at a set $X_1 = \{1\}$ with one element. Then $\mathcal{P}(X_1) = \{\phi, X_1\}$. So $E = \{\phi\}$ and $O = \{X_1\}$ so $|O| = |E| = 1$. Now consider $X_2 = \{1,2\}$. Here $\mathcal{P}(X_2) = \{\phi, \{1\}, \{2\}, X_2\}$ and $E = \{\phi, X_2\}$ and $O = \{\{1\}, \{2\}\}$ so $|O| = |E| = 2$. One more? Ok, let $X_3 = \{1,2,3\}$. Then $\mathcal{P}(X_3) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X_3\}$ has 8 elements and $E = \{\phi, \{1,2\}, \{1,3\}, \{2,3\}\}$ while $O = \{\{1\}, \{2\}, \{3\}, X_3\}$. Evidently, we can recursively define $E(n) = \begin{cases} 1 & \text{if } n = 1 \\ E(n-1) + 0(n-1) & \text{if } n > 1 \end{cases}$ since, all the elements of E with X_{n-1} are also elements of E with X_n and to every element of O for X_{n-1} we can add the new element of X_n and thereby get an element of E for X_n which was not one of those in E for X_{n-1}.

(b) Find a recurrence relation for $O(n)$ in termes of $O(n-1)$ and $E(n-1)$.

ANS: A very similar argement for that above shows that $O(n) = \begin{cases} 1 & \text{if } n = 1 \\ E(n-1) + 0(n-1) & \text{if } n > 1 \end{cases}$

(c) Since $|\mathcal{P}(X)| = 2^n$ it is easy to see that $|E| = |O| = 2^{n-1}$.