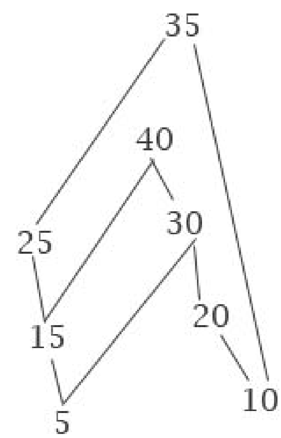
Math 15 – Discrete Structures – §2.5 – Homework 6 Solutions

<u>2.5#13</u>: Let X be the set of different nonzero monetary values (in U.S. or Canadian cents). In other words, $X \subseteq \mathbb{N}$. Define a relation $\vDash$on X as follows. $\forall a, b \in X, \vDash b$ if b can be obtained from a by adding a (possibly empty) collection of dimes (10 cents) and quarters (25 cents). So, for example, $25 \vDash 35$, but $25 \nvDash 30$.
Prove that $\vDash$ is a partial ordering on X.
ANS: We must satisfy the 3 properties of a poset.
i. Reflexivity: Since you can add an empty collection, every element is related to itself.
ii. Transitivity: If $a \vDash b$ then there are integers w and x such that $b = a + 10w + 25x$.
If $b \vDash c$ then there exist integers y and z such that $c = b + 10y + 25z$.
Substituting for b in this last equation, $c = (a + 10w + 25x) + 10y + 25z = a + 10(w + y) + 25(x + z)$.
Since the integers are closed under addition, this shows that $a \vDash c$.
iii. Antisymmetry: If $a \vDash b$ and $b \vDash a$ then the value of $a \leq b$ and vice versa, so $a = b$.

<u>2.5#14</u>: Let $X = \{5,10,15,20,25,30,35,40\}$, and let $\vDash$ be as in problem #13.
Draw the Hasse diagram, for the poset $(X, \vDash)$.
ANS: (a) (at right)



(b) The minimal elements are 5 and 10.

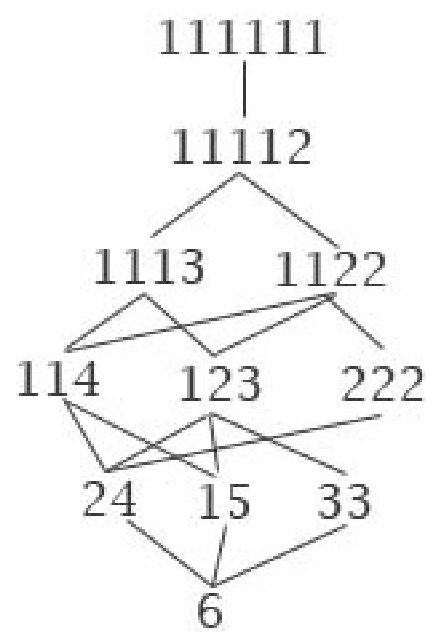
(c) Give a pair of incomparable elements in $(X, \vDash)$.
ANS: Clearly 35 and 40 are incomparable, since $35 \nvDash 40$ and $40 \nvDash 35$, but this is clearly true for any two values that differ by 5, or 15.

<u>2.5#18</u>: A *partition* of a positive integer *n* is a list of positive integers $a_1, a_2, \ldots, a_k$ such that $\sum_{i=1}^{k} a_i = n$. For example, the following are distinct partitions of 5.

5   1, 1, 1, 2   1, 2, 2   1, 1, 1, 1, 1

The order of the list doesn't matter, so 1,1,1,2 is the same as 1,2,1,1. There is a natural partial ordering of the set of partitions of *n*: if P_1 and P_2 are partitions, define $P_1 \preccurlyeq P_2$if P_1 can be obtained by combining parts of P_2. For example 1,2,2 $\preccurlyeq$ 1,1,1,2 since 1=1, 2=1+1, and 2=2. On the other hand, 1,4 and 2,3 are incomparable members of this poset.

(a) Write the partitions of 6 in a Hasse diagram.
ANS: (at right)



(b) Is this a total ordering? Why or why not?
ANS: It's very close, but not quite. 15 and 222 are incomparable.

2.5#22: Let B be the set of all four-digit binary strings, that is,

B = {0000, 0001, 0010, 0011, … , 1110, 1111}.
Define a relation $\lhd$ on B as follows: Let $x, y \in B$, where $x = x_1x_2x_3x_4$ and $y = y_1y_2y_3y_4$. We say that $x \lhd y$ if $x_i \leq y_i$ for i = 1,2,3,4. In other words, $x \lhd y$ if y has a 1 in every position where x does. So, for example, $0101 \lhd 0111$ and $0001 \lhd 1101$ but $0001 \ntriangleleft 1110$. The relation $\lhd$ is called the *bitwise* $\leq$.
Show that $(B, \lhd)$ is a poset.

ANS: We simply show that it satisfies the three conditions for a partially ordered set.

i. $\forall x \in B, x_i \leq x_i,$ so $x \lhd x$ and $\lhd$ is reflexive.

ii. $\forall x, y, z \in B,$ suppose $x \lhd y$ and $y \lhd z$ then $x_i \leq y_i \leq z_i \rightarrow x_i \leq z_i$ so $x \lhd z$ and $\lhd$ is transitive

iii. Suppose $\exists x, y \in B$ with $x \lhd y$ and $y \lhd x$. Then for all i, $x_i \leq y_i$ and $y_i \leq x_i$ so $x_i = y_i$ and x = y.