Math 15 – Discrete Structures – 2.3 Homework Solutions

<u>2.3#24</u>: Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined as $f(x) = \begin{cases} x+3 & \text{if } x \text{ is even} \\ x-5 & \text{if } x \text{ is odd} \end{cases}$

(a) Show that f is a one-to-one correspondence.
ANS: If x is even then there is some integer k such that $x = 2k$ and thus $f(x) = 2k + 3 = 2(k+1) + 1$ is odd.
If x is odd then there is some integer k such that $x = 2k + 1$ and thus $f(x) = 2k - 4 = 2(k-2)$ is even.
Suppose $f(x) = f(y)$ is odd. Then $x - 5 = y - 5 \Leftrightarrow x = y$. Now suppose $f(x) = f(y)$ is even. Then $x + 3 = y + 3 \Leftrightarrow x = y$.

(b) Find a formula for $f^{-1}(x)$.
ANS: $f^{-1}(x) = \begin{cases} x+5 & \text{if } x \text{ is even} \\ x-3 & \text{if } x \text{ is odd} \end{cases}$

<u>2.3#30</u>: Let $f: X \to Y$ and $g: Y \to Z$ be one-to-one correspondences. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
ANS: Let $z \in Z$. We need to show that $(g \circ f)\big((f^{-1} \circ g^{-1})(z)\big) = z$

$$\Leftrightarrow (g \circ f \circ f^{-1} \circ g^{-1})(z)$$
$$\Leftrightarrow (g \circ g^{-1})(z)$$
$$\Leftrightarrow 1z = z$$

<u>2.3#36</u>: Let $f: X \to Y$ be a function. For any subset $U \subset X$, define the set $f(U) = \{y \in Y | y = f(x) \text{ for some } x \in U\}$. In particular, $f(X)$ is the image of f.
(a) Let $A, B \in X$. Does $f(A \cap B) = f(A) \cap f(B)$ in general?
ANS: No, here's a counterexample. Let $f(x) = |x|$ and $A = [-2, -1]$, while $B = [1,2]$. Then $f(A \cap B) = \emptyset$ while $f(A) \cap f(B) = [1,2]$.

(b) Suppose $y \in f(A \cup B)$. Then $y = f(x)$ for some $x \in A \cup B$. Therefore $x \in A$ or $x \in B$, so $y \in f(A)$ or $y \in f(B)$. Thus $y \in f(A) \cup f(B)$.
Conversely, suppose $y \in f(A) \cup f(B)$. Then $y \in f(A)$ or $y \in f(B)$. So either there is some $x_1 \in A$ such that $y = f(x_1)$, or there is some $x_2 \in B$ such that $y = f(x_2)$. Either way, $x_i \in A \cup B$, so $y \in f(A \cup B)$.