Math 15 – Discrete Structures – 1.3 Homework Solutions

<u>1.3#12</u>: The domain for this problem is some unspecified collection of numbers. Consider the predicate $P(x, y) =$ "x is greater than y"

(a) Translate the following statement into predicate logic: "Every number has a number that is greater than it."
ANS: $(\forall x)(\exists y)(y > x)$
(b) Negate your expression from part (a), and simplify it so that no quantifier lies within the scope of a negation.
ANS: $\neg((\forall x)(\exists y)(y > x)) \Leftrightarrow (\exists x)(\forall y)(y \leq x)$

<u>1.3#14</u>: The domain of the following predicates is all integers greater than 1.

$P(x) =$ "x is prime."
$Q(x,y) =$ "x divides y."

Consider the following statement:

For every x that is not prime, there is some prime y that divides it.

(a) Write the statement in predicate logic.
ANS $(\forall x)\neg P(x)((\exists y)Q(y, x)$
(b) Formally negate the statement.
ANS: $\neg(\forall x \neg P(x)\big((\exists y)Q(y, x)\big) \Leftrightarrow (\exists x)P(x)\neg\big((\exists y)Q(y, x)\big) \Leftrightarrow (\exists x)P(x)(\forall y)\neg Q(y, x)$
(c) That is, there exists a prime that does not divide any number. Wrong, right?

<u>1.3#21</u>: Let the following predicates be given. The domain is all computer science classes.

$I(x) =$ "x is interesting."
$U(x) =$ "x is useful."
$H(x,y) =$ "x is harder than y."
$M(x,y) =$ "x has more students than y."

(a) Write the following statements in predicate logic.
i. All interesting CS classes are useful
ANS: $(\forall x)\big(I(x) \rightarrow U(x)\big)$
ii. There are some useful CS classes that are not interesting.
ANS: $(\exists x)(U(x) \wedge \quad \neg I(x))$
iii. Every interesting CS class has more students than any non-interesting CS class.
ANS $(\forall x)(\forall y)(I(x) \wedge \neg I(y)) \rightarrow M(x, y)$
(b) Write the following predicate logic statement in everyday English. Don't just give a word-for-word translation; your sentence should make sense. $(\exists x)[I(x) \wedge (\forall y)\big(H(x, y) \rightarrow M(y, x)\big)]$
ANS: There is a CS class that is interesting and has fewer students than all easier classes.
(c) Formally negate the statement from (b). Simplify your negation so that no quantifier lies withing the scope of a negation. State which derivation rules you are using.

Statement	reason
1. $\neg[(\exists x)\big[I(x) \wedge (\forall y)\big(H(x, y) \rightarrow M(y, x)\big)\big]]$	given
2. $(\forall x)\neg\big[I(x) \wedge (\forall y)\big(H(x, y) \rightarrow M(y, x)\big)\big]$	Existential negation
3. $(\forall x)\big[\neg I(x) \vee \neg[(\forall y)\big(H(x, y) \rightarrow M(y, x)\big)]\big]$	De Morgan
4. $(\forall x)\big[\neg I(x) \vee [(\exists y)\neg\big(H(x, y) \rightarrow M(y, x)\big)]\big]$	Universal negation
5. $(\forall x)\big[\neg I(x) \vee [(\exists y)\neg\big(\neg H(x, y) \vee M(y, x)\big)]\big]$	implication
6. $(\forall x)[\neg I(x) \vee [(\exists y)(H(x, y) \wedge \neg M(y, x))]]$	De Morgan

(d) Give a translation of the statement in everyday English.
ANS: Every CS class is either not interesting or there is an easier CS class that has fewer students.

<u>1.3#22</u>: Let the following predicates be given. The domain is all cars.

$$F(x) = \text{“}x \text{ is fast.”}$$
$$S(x) = \text{“}x \text{ is a sports car.”}$$
$$E(x) = \text{“}x \text{ is expensive.”}$$
$$A(x,y) = \text{“}x \text{ is safer than } y\text{.”}$$

(a) Write the following statements in predicate logic.
   i. All sports cars are fast. ANS: $(\forall x)(S(x) \rightarrow F(x))$
   ii. There are fast cars that aren't sports cars. ANS: $(\exists x)(F(x) \land \neg S(x))$
   iii. Every fast sports car is expensive. ANS: $(\forall x)\big(F(x) \land S(x)\big) \rightarrow E(x)$
(b) Write the following predicate logic statement in everyday English. Don't just give a word-for –word translation; your sentence should make sense. $(\forall x)[S(x) \rightarrow (\exists y)\big(E(y) \land A(y,x)\big)]$
ANS: All sports cars are more dangerous than some other car.
(c) Formally negate the statement from part (b). Simplify your negation so that no quantifier lies within the scope of a negation. State which derivation rules you are using.

Statement	reason
1. $\neg[(\forall x)[S(x) \rightarrow (\exists y)\big(E(y) \land A(y,x)\big)]]$	given
2. $(\exists x)\neg\big[S(x) \rightarrow (\exists y)\big(E(y) \land A(y,x)\big)\big]$	Universal negation
3. $(\exists x)\neg\big[\neg S(x) \lor (\exists y)\big(E(y) \land A(y,x)\big)\big]$	implication
4. $(\exists x)\big[S(x) \land \neg(\exists y)\big(E(y) \land A(y,x)\big)\big]$	De Morgan and double negative
5. $(\exists x)\big[S(x) \land (\forall y)\neg\big(E(y) \land A(y,x)\big)\big]$	Existential negation
6. $(\exists x)[S(x) \land (\forall y)(\neg E(y) \lor \neg A(y,x))]$	De Morgan
7. $(\exists x)[S(x) \land (\forall y)(E(y) \rightarrow \neg A(y,x))]$	implication

(d) There is a sports car that is at least as safe as every expensive car.

<u>1.3#23</u>: Domain = {0,1} with the predicate $P(x)$ defined by $P(0) = p$ and $P(1) = q$.
(a) Write $(\forall x)P(x)$ as a propositional logic formula using p and q.
ANS: $p \land q$, that is, P is always true.
(b) Write $(\exists x)(P(x))$ as a propositional logic statement
ANS: $p \lor q$, that is, P is sometimes true.
(c) In this situation, which derivation rule from propositional logic corresponds to the universal and existential negation rule of predicat logic?
ANS: De Morgan.

<u>1.3#24</u>: (a) Give an example of a pair of predicates $P(x)$ and $Q(x)$ in some domain to show that the $\exists$ quantifier does not distribute over the $\land$ connective. That is, give an example to show that

$$(\exists x)\big(P(x) \land Q(x)\big) \neq (\exists x)(P(x) \land Q(x))$$

ANS: In the domain of real numbers, let $P(x)$ denote “x is non-negative” and let $Q(x)$ denote “x is negative.” The statement that $(\exists x)\big(P(x) \land Q(x)\big)$ is false: no number can be both positive <u>and</u> negative. However the statement that $(\exists x)\big(P(x)\big) \land (\exists x)(Q(x))$ is true: there are some numbers that are positive <u>and</u> there are some that are negative.

(b) And also show that $(\exists x)\big(P(x) \vee Q(x)\big) = (\exists x)\big(P(x)\big) \vee (\exists x)(Q(x))$

ANS: Saying that there is a number that is positive or negative is the same as saying that there is a positive number or there is a negative number.

<u>1.3#25</u>: (a) Give an example to show that $\forall$ does not distribute over $\vee$.
ANS: In the domain of real numbers let $Q(x)$ = "x is a rational number" and let $I(x)$ = "x is an irrational number." Then $(\forall x)(Q(x) \vee I(x))$ is true but $(\forall x)Q(x) \vee (\forall x)I(x)$ is not true since it is neither true that every real is rational, nor is it true that every real is irrational.

(b) It is a fact that $\forall$ distributes over $\wedge$. Check that your example from part (a) satisfies this equivalence rule.
ANS: $(\forall x)(Q(x) \wedge I(x))$ is never true and $(\forall x)Q(x) \wedge (\forall x)I(x)$ is not true since not all reals are rational and neither are all reals irrational, so 0 and 0 = 0.