Math 15 – Discrete Structures – §4.6 – Homework 17 Solutions

<u>4.6#16</u>: Prove by contradiction: $2^n \notin \Omega(n!)$.
SOLN: Suppose that $2^n \in \Omega(n!)$. Then, by definition, there exist K and N such that $n \geq N$ will assure that $2^n \geq K \cdot n!$. But $2^n \geq K \cdot n! \Leftrightarrow \frac{1}{K} \geq \frac{n!}{2^n} = \frac{1}{2} \cdot \frac{2}{2} \cdot \frac{3}{2} \cdots \frac{n}{2} \geq \frac{n}{4}$, for all n , a contradiction.

<u>4.6#19</u>: Prove that if $f: \mathbb{N} \to \mathbb{R}^+$and $k > 0$, then $kf(n) \in \Theta\big(f(n)\big)$.
Proof: This is kind of obvious: just take $K_1 = K_2 = k$. Then $K_1 f(n) \leq kf(n) \leq K_2 f(n)$.

<u>4.6#20</u>: Prove that if $f(n)$ is a polynomial of degree p with all non-negative coefficients, then $f \in \Theta(n^p)$.
Proof: Since $n \geq 1$,

$$(a_0 + a_1 + \cdots + a_{p-1})\left(\frac{1}{a_p}\right)a_p n^p = (a_0 + a_1 + \cdots + a_{p-1})n^{p-1} \geq a_0 + a_1 n + \cdots + a_{p-1}n^{p-1}$$

This shows that $a_0 + a_1 n + \cdots + a_{p-1}n^{p-1} \in \mathcal{O}(a_p n^p)$. We know that if $g(n) \in \mathcal{O}(f(n))$ then $f(n) + g(n) \in \mathcal{O}\big(f(n)\big)$ so we conclude that $a_0 + a_1 n + \cdots + a_{p-1}n^{p-1} + a_p n^p \in \mathcal{O}(a_p n^p)$. Now by problem #19 above, we are done.

<u>4.6#24</u>: An urn contains *n* distinct balls.
(a) Give a big-Θ estimate for the number of ways to draw a sequence of 3 balls with replacement.
SOLN: Since the number of ways to do this is $n(n)(n)$ which is polynomial of degree 3, this is $\Theta(n^3)$
(b) Give a big-Θ estimate for the number of ways to draw a sequence of 3 balls without replacement.
SOLN: Since the number of ways to do this is $n(n-1)(n-2)$ which is polynomial of degree 3, this is $\Theta(n^3)$
(c) Give a big-Θ estimate for the number of ways to draw a sequence of *n* balls without replacement.
SOLN: This is exactly $n! \in \Theta(n!)$.

<u>4.6#28</u>: A certain algorithm processes a list of *n* elements. Suppose that `Subroutine`$_a$ requires $n^2 + 2n$ operations and `subroutine`$_b$ requires $3n^3 + 7$ operations. Give a big-Θ estimate for the number of operations performed by the following pseudocode segment.

```
for i∈ {1,2, … , n} do
    Subroutine_a
    Subroutine_b
```

SOLN: `Subroutine`$_b$ requires the higher order number of operations, so we can ignore `Subroutine`$_a$. Occurs *n* times, the pseudocode segment requires $n(3n^3 + 7) \in \Theta(n^4)$ operations.

<u>4.6#30</u>: Prove that if $f, g: \mathbb{N} \to \mathbb{R}^+$ then $f \in \mathcal{O}(g) \Leftrightarrow g \in \mathcal{O}(f)$.
Proof: Suppose $f \in \mathcal{O}(g)$. Then there exist positive numbers *K* and *N* such that $f(n) \leq Kg(n)$ for all $n \geq N$. Let $K' = 1/K$. Then for all $n \geq N$, $g(n) \geq K'f(n)$, so $g \in \Theta(f)$.
Similarly, if $g \in \mathcal{O}(f)$. Then there exist positive numbers *K* and *N* such that $g(n) \leq Kf(n)$ for all $n \geq N$. Let $K' = 1/K$. Then for all $n \geq N$, $f(n) \geq K'g(n)$, so $f \in \Theta(g)$.