Math 15 – Discrete Structures – §4.4 – Homework 15 Solutions

<u>4.4#6</u>: If you roll to four-sided dice (faces numbered 1, 2, 3, and 4), what is the probability of rolling a 5?
ANS: There are $4^2 = 16$ outcomes, four of which comprise a five: (1,4), (2,3), (3,2) and (4,1). Thus the probability of a 5 is $\frac{4}{16} = \frac{1}{4}$.

<u>4.4#10</u>: In a suitable font, the letters A, H, I, M, O, T, U, V, W, X, Y are all mirror images of themselves. A string made from these letters will be a mirror image of itself if it reads the same backward as forward: for isample MOM, YUMMUY, MOTHTOM. If a four-letter string in these letters is chosen at random, what is the probability that this string is a mirror image of itself?
ANS: There are $11^4 = 14641$ possible 4-letter words with this choice of 11 letters. There are $11^2 = 121$ different ways to choose the first two letters and then only one way to choose symmetric letters. Thus the probability of choosing a mirror image word is $\frac{121}{14641} = \frac{1}{121}$.

<u>4.4#14</u>: In a class of 17 students, 3 are math majors. A group of four students is chosen at random.
(a) What is the probability that the group has no math majors?
ANS: There are $\binom{17}{4} = \frac{17\cdot16\cdot15\cdot14}{4\cdot3\cdot2} = 2380$ different groups of 4 students from the class. $\binom{14}{4} = \frac{14\cdot13\cdot12\cdot11}{4\cdot3\cdot2} = 1001$ of these have no math major. So the probability that a randomly chosen group has no math major is $\frac{14\cdot13\cdot12\cdot11}{4\cdot3\cdot2} \div \frac{4\cdot3\cdot2}{17\cdot16\cdot15\cdot14} = \frac{14}{17}\cdot\frac{13}{16}\cdot\frac{12}{15}\cdot\frac{11}{14} = \frac{143}{340} \approx 0.42$ or 42% probability.

(b) What is the probability that the group has at least one math major?
ANS: This is the sum of the probabilities that there is 1, 2, or 3. It's easier to compute the probability that there is not zero, which is $\Pr(n \geq 1) = 1 - \Pr(n = 0) = 1 - \frac{143}{340} = \frac{197}{340} \approx 0.58$ or 58%

(c) What is the probability that the group has exactly 2 math majors?
ANS: There are 3 ways to choose 2 of the 3 math majors. For each of these, there are $\binom{14}{2} = 91$ ways to choose the other 2 members from the class, so the probability of choosing exactly 2 math majors is $\frac{3\cdot91}{2380} = \frac{39}{340}$.

<u>4.4#16</u>: A game warden catches 10 fish in the lake, marks them and then returns them to the lake. Three weeks later, the warden catches five fish, and discovers that two of them are marked.
(a) Let *k* be the number of fish in the lake. Find the probability (in terms of *k*) that two of the five randomly selected fish are marked.
ANS: There are $\binom{k}{5}$ ways to choose 5 fish from the lake. There are $\binom{10}{2} = 45$ ways to choose which 2 of the 10 marked fish are chosen and then $\binom{k-10}{3}$ ways to choose the other 3 fish from among the unmarked fish. Thus the probability of choosing exactly 2 marked fish is

$$\frac{45\binom{k-10}{3}}{\binom{k}{5}} = \frac{45(k-10)(k-11)(k-12)5\cdot4\cdot3\cdot2}{k(k-1)(k-2)(k-3)(k-4)3\cdot2} = \frac{900(k-10)(k-11)(k-12)}{k(k-1)(k-2)(k-3)(k-4)}$$

(b) What value of *k* will maximize this probability?
ANS: Entering the formula =45*COMBIN(A2-10,3)/COMBIN(A2,5) into the cell B2 in the Excel spreadsheet as shown below and then copying this into the cells below it give the probabilities for *k* = 13, 14, … , 32 and demonstrate that the maximum probability occurs for *n* = 24 or 25 where the probability is $\frac{3083}{8000} = 0.385375$

=45*COMBIN(A2-10,3)/COMBIN(A2,5)

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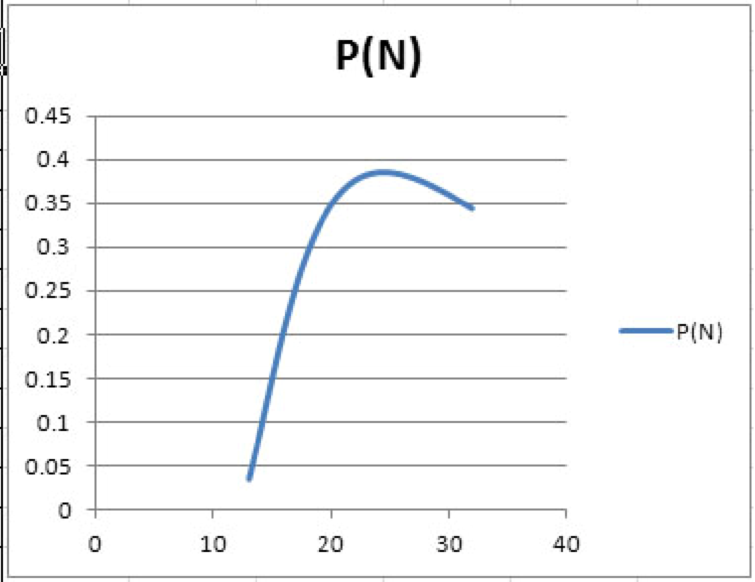
	A	B
1	N	P(N)
2	13	0.034965
3	14	0.08991
4	15	0.14985
5	16	0.206044
6	17	0.254525
7	18	0.294118
8	19	0.325077
9	20	0.348297
10	21	0.364883
11	22	0.37594
12	23	0.382478
13	24	0.385375
14	25	0.385375
15	26	0.383095
16	27	0.379041
17	28	0.373626
18	29	0.367185
19	30	0.359985
20	31	0.352243
21	32	0.344132



4.4#18: Ten cards are numbered 1 through 10. The cards are shuffled thoroughly and placed in a stack. What is the probability that the numbers on the top three cards are in ascending order?
ANS: There are 3! = 6 ways to put the top three cards in order and only one of these is ascending, so the probability is 1/6.

4.4#22: An urn contains three red balls, four white balls and two black balls. Three balls are drawn from the urn at random without replacement. For each red ball drawn, you win \$10, and for each black ball drawn, you lose \$15. Let X represent your net winnings.
(a) Compute $\Pr(X = 0)$.
ANS: There is only one way this could happen: pick 3 white balls (4 ways to do that). Since there are $\binom{9}{3} = \frac{9\cdot 8\cdot 7}{3\cdot 2} = 84$ ways to pick 3 of 9 balls, $\Pr(X = 0) = \frac{4}{84} = \frac{1}{21}$
(b) Compute $\Pr(X < 0)$.

ANS: $\Pr(X < 0) = \Pr(2\text{ black}) + \Pr(1\text{ black and 1red}) + \Pr(1\text{ black and 0 Red}) = \frac{7+2\cdot 3\cdot 4+2\cdot\binom{4}{2}}{84} = \frac{43}{84}$

(c) Compute $E(X)$, your expected winnings.
ANS: The possible values of X are $-30, -20, -15, -5, 5, 10, 20, 30$ achieve by, respectively, 2 black and 1 white, 2 black and 1 read, 1 black and 2 white, 1 black and 1 red, 1 black and 2 red, 1 red and 2 white, 2 red and 1 white or 3 red. The expectation is thus
$-30\Pr(2B, 1W) - 20\Pr(2B, 1R) - 15\Pr(1B, 2W) - 5\Pr(1B, 1R) + 5\Pr(1B, 2R) + 10\Pr(1R, 2W) +$

$20\,\Pr(2R, 1W) + 30Pr(3R) = -30\left(\frac{4}{84}\right) - 20\left(\frac{3}{84}\right) - 15\left(\frac{2\cdot 6}{84}\right) - 5\left(\frac{2\cdot 3}{84}\right) + 5\left(\frac{2\cdot 3}{84}\right) + 10\left(\frac{3\cdot 6}{84}\right) + 20\left(\frac{3\cdot 4}{84}\right) + 30\left(\frac{1}{84}\right) = \frac{-120-60-180-30+180+240+30}{84} = \frac{60}{84} = \$\frac{5}{7}.$

4.4#26: Conduct a random experiment of flipping a coin five times. Let X be the number of heads.

(a) Compute $\Pr(X > 3)$

ANS: $\Pr(X > 3) = \Pr(X = 4) + \Pr(X = 5) = \frac{5}{32} + \frac{1}{32} = \frac{3}{16}$

(b) Compute $E(X) = 0 + 1\cdot\frac{5}{32} + 2\cdot\frac{10}{32} + 3\cdot\frac{10}{32} + 4\cdot\frac{5}{32} + 5\cdot\frac{1}{32} = \frac{5+20+30+20+5}{32} = \frac{80}{32} = \frac{5}{2}$, as you might expect.