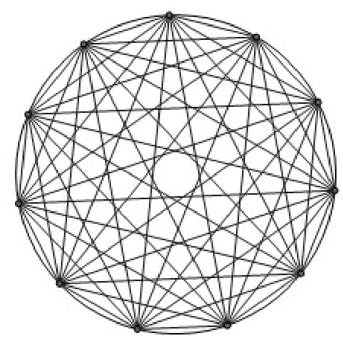
Math 15 – Discrete Structures – §4.3 – Homework 14 Solutions

<u>4.3#8</u>: Let S be a set of n numbers. Let X be the set of all subsets of size k, and let Y be the set of all ordered k-tuples $(s_1, s_2, \dots, s_k)$ such that $s_1 < s_2 < \cdots < s_k$. That is,
$X = \{\{s_1, s_2, \dots, s_k\} | s_i \in S \text{ and all the } s_i's \text{ are distinct}\}$, and
$Y = \{(s_1, s_2, \dots, s_k) | s_i \in S \text{ and } s_1 < s_2 < \cdots < s_k\}$.
(a) Define a one-to-one correspondence $f: X \to Y$. Explain why f is one-to-one and onto.
ANS: Define $f(\{s_1, s_2, \dots, s_k\})$ as the k-tuple that puts the elements of the set in order from smallest to largest. The function is well-defined because there is only one way to put the elements in order. It's 1-1 and onto because given any k-tuple $(s_1, s_2, \dots, s_k)$, there's exactly one set that it could have come from.

(b) Determine $|X|$ and $|Y|$.
ANS: There are $\binom{n}{k}$ ways to choose k elements from n, and this is the number of elements in both X and Y.

<u>4.3#12</u>: How many ways are there to rearrange the letters in INANENESS?
ANS: There are 9! ways to arrange the 9 letters but, due to the 3 identical N's and the pairs of E's and S's, only $\frac{9!}{3!\cdot 2\cdot 2} = 15120$ arrangements which are distinct.

<u>4.3#14</u>: Count the points of intersection shown in the picture, not counting the points that lie on the circle.



ANS: There is a 1-1 correspondence between the number of points of intersection on the interior and the number of ways of choosing 4 of the 11 points on the circle. To see this note that for every choice of 4 points there is a cyclic quadrilateral and this quadrilateral determines exactly one point of intersection, the point of intersection of its diagonals.
Thus there are $\binom{11}{4} = \frac{11\cdot 10\cdot 9\cdot 8}{4\cdot 3\cdot 2} = 330$ points.

<u>4.3#22</u>: On the eve of an election, a radio station is forced to play 20 campaign ads in a row. Of these 20 ads, 15 are for the Tory candidate and 5 are for the Labour candidate. Prove that the station must play at least 3 Tory ads in a row at some point. Use the generalize pigeon hole principle to justify your answer.
ANS: Think of the "pigeons" as the 15 Tory ads and the "holes" as the 6 groups that the 5 Labour ads divide these into. The generalized pigeon hole principle says there are at least $\left\lceil \frac{15}{6} \right\rceil = 3$ Tory ads in the largest group.

<u>4.3#29</u>: Use the pigeon hole principle to explain why every rational number has a decimal expansion that either terminates or repeats. In the case where a rational number m/n has a repeating decimal expansion, find an upper bound (in terms of the divisor, n) on the number of digits in the part that repeats.
ANS: According the standard division algorithm, at each iteration we find the largest multiple of the divisor which is less than the current remainder (multiplying the remainders by 10 at each iteration) and record this multiple in the decimal expansion. At each stage the remainder must be a non-negative integer less than the divisor. Since there are only n such remainders: 0, 1, 2, … , n – 1, the pigeon hole principle says that eventually we must repeat a remainder, in which case the whole process will cycle.

<u>4.3#30</u>: Suppose that 100 lottery tickets are given out in sequence to the first 100 guests to arrive at a party. Of these 100 tickets, only 12 are winning tickets. The generalized pigeon hole principle says there must be a streak of at least l losing tickets in a row. Find l.
ANS: Think of the 88 losing tickets as the pigeons and these are to be separated into 13 groups by the 12 winning tickets. The GPH says there must be a streak of at least $\left\lceil \frac{88}{13} \right\rceil = 7$ losing tickets in a row.