Math 15 – Discrete Structures – §3.4 – Homework 11 Solutions

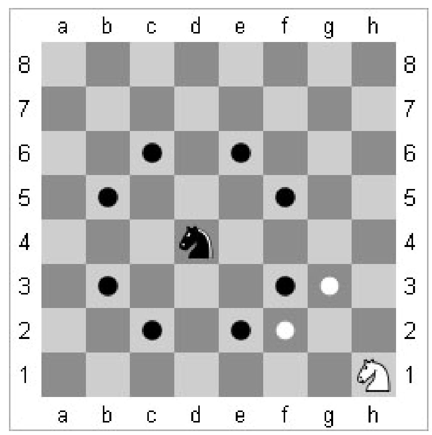
<u>3.4#18</u>: Prove that $K(n)$, the nth term in the sequence of shapes whose limit is the Koch snowflake fractal has perimeter $3 \cdot \left(\frac{4}{3}\right)^{n-1}$, where the equilateral triangle in $K(1)$ has a side length of 1 unit.

ASN: Proof by induction on n: $K(1)$ is an equilateral triangle with side length 1, so its perimeter is $3 = 3 \cdot \left(\frac{4}{3}\right)^{1-1} = 3 \cdot 1$. Now suppose that $K(n-1) = 3 \cdot \left(\frac{4}{3}\right)^{n-1-1} = 3 \cdot \left(\frac{4}{3}\right)^{n-2}$ for some $n > 1$. To form the next iteration, $K(n)$, involves replacing the middle third with two segment each of the same length as the middle third, thus increasing the length of the perimeter by a factor of $\frac{4}{3}$. Thus the length of the perimeter of $K(n) = \frac{4}{3} \cdot K(n-1) = \frac{4}{3} \cdot 3 \cdot \left(\frac{4}{3}\right)^{n-2} = 3 \cdot \left(\frac{4}{3}\right)^{n-1}$. QED.

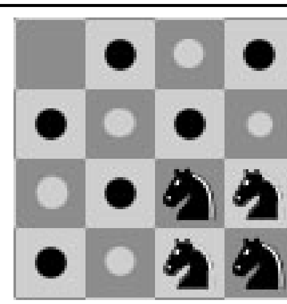
<u>3.4#22</u>: Define X recursively as follows:
B: 3 and 7 are in X.
R: If $x, y \in X$ then $x + y \in X$.
Prove that $\forall n \in \mathbb{N}, n \in X$.

ANS: 12 = 3 + 3 + 3 + 3 is in X. 13 = 7 + 3 + 3 is in X and 14 = 7 + 7 is in X. To get any number higher than 14 we can simply repeatedly add 3 to one of 12, 13, or 14 since that comprises the equivalence classes modulo 3 of all such integers.

<u>3.4#24</u>: In the game of chess, a knight moves by jumping to a square that is 2 units away in one direction and one unit away in the other direction. The diagram at right indicates that a black knight at d4 can move to 8 other positions in one move, and the white horse at h1 can only move to two other positions. Prove by induction that a knight can move from any square to any other square on an $n \times n$ chessboard via a sequence of moves, for all $n \geq 4$.



ANS: On a 4×4 chessboard a knight can get to any position from any other position. To see this, note that a knight can get from any position in the lower right hand four squares (where the knights are shown in the second illustration to the right) to any other one by a sequence of 3 or 4 moves and that all the circles in the diagram are Are reachable from one of the one of the lower right knight positions shown in one move. Also that the upper left corner is attainable from the lower right corner in two moves.



The inductive hypothesis that there is a sequence of knight moves to get from any square to any other square on a $(k-1) \times (k-1)$ chessboard, for some $k > 4$. A $k \times k$ chessboard can be viewed as having four overlapping $(k-1) \times (k-1)$ boards inside of it.

Note that there is no way to visit each square on a 4×4 chess board exactly once, so there is no "knight's tour" of a 4×4 chess board. Is there a knight's tour of a $n \times n$ chess board for some $n > 4$?