Math 15 – Discrete Structures – 1.1 & 1.2 Homework Solutions

<u>1.1#22</u>: Mathematicians say that "Statement P is a *sufficient condition* for statement Q" if $P \rightarrow Q$ is true. In other words, in order to know that Q is true, it is sufficient to know that P is true. Let x be an integer. Give a sufficient condition on x for $x/2$ to be an integer.

ANS: If there exists an integer k such that $x = 16k$, then $x/2$ is an integer. (Note the condition is not necessary, but it is sufficient.)

<u>1.1#24</u>: Let Q be a quadrilateral. Give a sufficient but not necessary condition for Q to be a parallelogram.

ANS: Q is a square is over-determined, so we could relax that to either Q is a rhombus or Q is a rectangle is still over-determined, that is, sufficient, but not necessary.

<u>1.1#28</u>: The NAND connective is defined as $p \uparrow q = \neg(p \wedge q)$. To see this in a truth table:

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$p \uparrow q$
0	0	1	1	1	0	1	1
0	1	1	0	1	0	1	1
1	0	0	1	1	0	1	1
1	1	0	0	0	1	0	0

<u>1.2#19</u>: Write a proof sequence to establish that $p \Leftrightarrow p \vee p$ is a tautology.

ANS: We need to establish both $p \Rightarrow p \wedge p$ and $p \Leftarrow p \wedge p$.
First $p \Rightarrow p \wedge p$

Statement	Reason
p	Given
$p \wedge p$	Conjunction

And then $p \Leftarrow p \wedge p$.

Statement	Reason
$p \wedge p$	Given
p	Simplification

<u>1.2#20</u>: Write a proof sequence to establish that $p \Leftrightarrow p \vee p$ is a tautology.

ANS: We need to establish both $p \Rightarrow p \vee p$ and $p \Leftarrow p \vee p$.
First $p \Rightarrow p \vee p$

Statement	Reason
p	**Given**
$p \vee p$	addition

And then $p \Leftarrow p \wedge p$.

Statement	Reason
$p \vee p$	Given
$\neg\neg(p \vee p)$	Double negation
$\neg(\neg p \wedge \neg p)$	De Morgan
$\neg(\neg p)$	Simplification
p	double negation

1.2#24: Use a truth table to show that

$$\left.\begin{matrix} p \to q \\ \neg p \end{matrix}\right\} \Rightarrow \neg q$$

Is not a tautology. (This example shows that substitution isn't valid for inference rules, in general. Substituting the weaker statement q for the stronger statement, p, in the expression $\neg p$ doesn't work.)

p	q	$p \to q$	$\neg p$	$\neg q$	$(p \to q) \land \neg p$	$(p \to q) \land \neg p \to \neg q$
0	0	1	1	1	1	1
1	0	0	0	1	0	1
0	1	1	1	0	1	0
1	1	1	0	0	0	1

So $\neg q$ may be true, but if $(p \to q) \land \neg p$ is false, then $p \to q$ is false. If this were a tautology, the last column would be all true.

1.2#25: (a) Fill in the reasons in the following proof sequence. Make sure you indicate which step(s) each derivation rule refers to

Statement	Reason
1. $p \to (q \to r)$	Given
2. $\neg p \lor (q \to r)$	Implication (1)
3. $\neg p \lor (\neg q \lor r)$	Implication (2)
4. $(\neg p \lor \neg q) \lor r$	Associativity
5. $\neg(p \land q) \lor r$	De Morgan
6. $(p \land q) \to r$	Implication (5)

(b) Explain why the proof above is reversible.
ANS: Every step is justified by an equivalence.

(c) The proof in part (a) (along with its reverse establishes the following tautology:

$$p \to (q \to r) \Leftrightarrow (p \land q) \to r$$

Therefore, to prove an assertion of the form $A \Rightarrow B \to C$ it is sufficient to prove

$$\left.\begin{matrix} A \\ B \end{matrix}\right\} \Rightarrow C$$

Instead. Use this to rewrite the tautology

$$p \land (q \to r) \Rightarrow q \to (p \land r)$$

As a tautology of the form

$$\left.\begin{matrix} A \\ B \end{matrix}\right\} \Rightarrow C$$

Where C does not contain the $\to$ connective. (The process of rewriting a tautology this way is called the *deduction method*.)

ANS: This seems a little slippery. The answer in the back of the book seems to make sense:

$$\left.\begin{matrix} p \land (q \to r) \\ q \end{matrix}\right\} \Rightarrow p \land r$$

Which is read, "Given $p \wedge (q \rightarrow r)$ and given q, therefore p and r." Certainly, we are assuming p together with the implication that $q \rightarrow r$ and q so also r, thus p and r.

1.2#26: This exercise will lead you through a proof of the *distributive property* of ∧ over ∨.
We will prove:

$$p \wedge (q \vee r) \Longrightarrow (p \wedge q) \vee (p \wedge r)$$

(a) The above assertion is the same as the following:

$$p \wedge (q \vee r) \Longrightarrow \neg(p \wedge q) \rightarrow (p \wedge r)$$

Why?
ANS: The right hand sides of these inferences are the same because of the implication rule.

(b) Use the deduction rule from Exercise 25(c) to rewrite the tautology from part (a).
ANS: We have $p \wedge (q \vee r) \Longrightarrow \neg(p \wedge q) \rightarrow (p \wedge r)$
Let *A* be the statement $p \wedge (q \vee r)$
Let *B* be the statement $\neg(p \wedge q)$
Let *C* be the statement $p \wedge r$
Then the inference looks like this:

$$\left.\begin{matrix} p \wedge (q \vee r) \\ \neg(p \wedge q) \end{matrix}\right\} \Longrightarrow p \wedge r$$

(c) Prove your written tautology.
ANS: A truth table is not too onerous here:

p	q	r	$q \vee r$	$p \wedge q$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Here's a proof – note that the proof seems significantly more difficult than the truth table in this case:

Statement	Reason
1. $p \wedge (q \vee r)$	Given
2. $\neg(p \wedge q)$	Given
3. $\neg p \vee \neg q$	De Morgan
4. $\neg q \vee \neg p$	Commutivity
5. $q \rightarrow \neg p$	Implication
6. p	Simplification, 1
7. $\neg(\neg p)$	Double negative
8. $\neg q$	Modus Tollens 5,7
9. $q \vee r$	Simplification, 1
10. $r \vee q$	Commutivity
11. $\neg(\neg r) \vee q$	Double negative
12. $\neg r \rightarrow q$	Implication
13. $\neg(\neg r)$	Modus tollens 8, 12
14. r	Double negation
15. $p \wedge r$	Conjunction 6, 14