Math 15 - Spring 2017 - Homework 5.3 Solutions

1. (5.3 # 18) Consider the following algorithm for finding a target element t in an array $\{x_1, x_2, \ldots, x_n\}$.

   Assume as a precondition that exactly one of the x_i's is equal to t.

   Choose i at random from $\{1, 2, \ldots, n\}$.
   `while` $x_i \neq t$ `do`
       `Choose` i `at random from` $\{1, 2, \ldots, n\}$.
   `print Element` t was found in location i.

   This algorithm continues to guess randomly until it finds where t is. Note that it does not keep track of its previous guesses, so it may check the same location more than once.

   (a) Find the best-case number of $\neq$ comparisons made by this algorithm.
   ANS: What luck! There it is: first in line, for a grand total of 1 comparison.

   (b) Warning: Tricky. Find the worst-case number of $\neq$ comparisons made by this algorithm.
   ANS: Worst case is that you guess wrong every time for an eternity: ∞

   (c) Calculus required. Use the sum of an infinite series to find the average case number of $\neq$ comparisons made by this algorithm.
   ANS: The average number of comparisons is the expected value, $E(X) = \sum_{i=1}^{\infty} n \cdot P(X = n)$, where $P(X - n)$ is the probability that the first correct guess is the nth guess; that is, $n-1$ independently wrong guesses, each with a probability of $\frac{n-1}{n}$, followed by a right guess, with a probability of $\frac{1}{n}$.
   Thus $P(X = i) = \left(\frac{n-1}{n}\right)^{i-1} \frac{1}{n}$ and $E(X) = \sum_{i=1}^{\infty} i \cdot \left(\frac{n-1}{n}\right)^{i-1} \frac{1}{n} = \frac{1}{n} + 2 \cdot \frac{n-1}{n^2} + 3 \cdot \frac{(n-1)^2}{n^3} + \cdots$
   Let's focus on this sum and simplify it a bit. We can bring out the factor of $\frac{1}{n}$ and then reindex the sum to start at $i = 0$: $E(X) = \frac{1}{n} \sum_{i=0}^{\infty} (i+1) \cdot \left(\frac{n-1}{n}\right)^{i}$ Now...looks like the power rule of differentiation in calculus has been acting on this thing: $E(X) = \frac{1}{n} \sum_{i=0}^{\infty} n^2 \cdot \frac{d}{dn} \left(\frac{n-1}{n}\right)^{i+1}$
   Now wave your hands furiously enough so you can reverse the order of summation and differentiation on this infinite sum to get $E(X) = n \cdot \frac{d}{dn} \sum_{i=0}^{\infty} \left(\frac{n-1}{n}\right)^{i+1} = n \frac{d}{dn} \sum_{i=0}^{\infty} r^i$ is geometric, with $r = 1 - \frac{1}{n}$, so the series converges to $n \cdot \frac{d}{dn} \frac{1}{1-r} = n \cdot \frac{d}{dn} n = n$. Obviously.

2. (5.3 # 20) The concept of best-, worst-, and average-case analyses extends beyond algorithms to other counting problems in mathematics. Recall that the height of a binary tree is the number of edges in the longest path from the root to a leaf.

   (a) Find the best-case height of a binary tree with five nodes.
   ANS: Best case is the depth of $\lfloor \log_2(5) \rfloor = 2$.

   (b) Find the worst-case height of a binary tree with five nodes.
   ANS: The worst case is when the tree has only one branch: $n - 1 = 4$.

   (c) Find the average-case height of a binary tree with five nodes. For this problem, you will have to list all possible binary trees with five nodes. Assume that each of these is equally likely to occur.
   ANS: In the Sage reference, complete binary trees, there may be a way to generate these...good searching!

The number of such trees is well-known as the n^{th} Catalan number): $\dfrac{(2n)!}{(n+1)!n!} = \dfrac{C(2n,n)}{n+1}$ and the fifth Catalan number is $42 = 6+20+16$ so the average case is $(2\cdot 6+3\cdot 20+4\cdot 16)/42 = \dfrac{68}{21} = 3.\overline{238095}$

(d) Find the worst-case height of a binary tree with n nodes.
ANS: $n-1$

(e) Approximate the best-case height of a binary tree with n nodes.
ANS: $\lfloor \log_2(n) \rfloor$.

3. (5.3 # 22) Find the best-, worst-, and average-case values for rolling two standard six-sided dice. (Hint: Refer to Example 4.36.)
Best: 2. Worst: 12. Average:
$\frac{1}{36}\cdot 2+\frac{2}{36}\cdot 3+\frac{3}{36}\cdot 4+\frac{4}{36}\cdot 5+\frac{5}{36}\cdot 6+\frac{6}{36}\cdot 7+\frac{5}{36}\cdot 8+\frac{4}{36}\cdot 9+\frac{3}{36}\cdot 10+\frac{2}{36}\cdot 11+\frac{1}{36}\cdot 12 = 7.$