Math 15 - Spring 2017 - Homework 4.2 Solutions

1. (4.3 # 24) Let G be a simple graph with two or more vertices. Prove that there is a pair of vertices in G having the same degree.
   Proof (induction on the number of vertices) . Since, in a simple graph, a vertex can have at most $n-1$ edges, so the degree of a vertex is $\leq n-1$. If there is a vertex of degree 0 if and only if there is no vertex of degree $n-1$. So there are $n-1$ possible values for the degrees of the vertices of G; but there are n verticess, so, by the pigeon hole principle, two vertices must have the same degree

2. (4.3 # 28) A small college offers 250 different classes. No two classes can meet at the same time in the same room, of course. There are 12 different time slots at which classes can occur. What is the minimum number of classrooms needed to accommodate all the classes? ANS: In the pigeon hole metaphor, think of the "pigeons" be the 250 classes, and the "holes" as the 12 times. The minimum number of classrooms is then $\left\lceil \frac{250}{12} \right\rceil = 21$.

3. (4.3 # 30) Suppose that 100 lottery tickets are given out in sequence to the first 100 guests to arrive at a party. Of these 100 tickets, only 12 are winning tickets. The generalized pigeonhole principle guarantees that there must be a streak of at least l losing tickets in a row. Find l. ANS: In this case think of 88 losing tickets as the pigeons and 13 intervals between drawing of a winning ticket as the holes. Then there must be an interval containing $\left\lceil \frac{88}{13} \right\rceil = 7$ is the minimum number of losing tickets in a row. You can also just try to spread the winning tickets as thinly as possible...so you have 7 losing tickets followed by a winning ticket, then again 7 losing tickets followed by a winning ticket, and so on in in 12 groups, acounting for 96 tickets, and then the remaining 4 losing tickets at the end: LLLLLLLWLLLLLLLWLLLLLLLWLLLLLLLWLLLLLLLWLLLLLLLWLLLLLLLWLLLLLLLWLLLLLLLWLLLLLLLWLLLLLLLWLLLLLLLWLLLL There's nowhere to put those extra 4 tickets without increasing a streak of losing tickets.

4. (4.3 # 32) Show that if the edges of the complete graph on eight vertices are colored red and green, then there is either a three-circuit or a four-circuit whose edges are all the same color. ANS: Pick a vertex v. There are seven edges on v, so by the generalized pigeonhole principle, four of these edges e_1, e_2, e_3, e_4 must be the same color (say green, without loss of generality). Let the other vertices of edges $e_1, e_2, e_3 m e_4$ be w, x, y, z, respectively. If the complete subgraph formed by w, x, y, z has all red edges, we are done. But if one of the edges is green, then it forms a green triangular circuit with v.

   Seems like a stronger result could be established. If there necessarily a complete subgraph k_4 which is monochromatic?