Math 15 - Spring 2017 - Homework 3.2 Solutions

1. (3.2 # 10 (not assigned)) Consider the following recurrence relation:

$$P(n) = \begin{cases} 1 & : n = 0 \\ P(n-1) + n^2 & : n > 0 \end{cases}$$

(a) Compute the first eight values of $P(n)$.
ANS: See below.

(b) Analyze the sequences of differences. What does this suggest about the closed-form solution?

ANS:

n	$P(n)$	ΔP	$\Delta^2 P$	$\Delta^3 P$
0	1	-	-	-
1	2	1	-	-
2	6	4	3	-
3	15	9	5	2
4	31	16	7	2
5	56	25	9	2
6	92	36	11	2
7	141	49	13	2
8	205	64	15	2

Since the third order differences are constant, we can fit a polynomial model of degree 3: $P(n) = a_3 n^3 + a_2 n^2 + a_1 n + a_0$. A good candidate will have $P(0) = a_0 = 1$ and $P(1) = a_3 + a_2 + a_1 + 1 = 2$, $P(2) = 8a_3 + 4a_2 + 2a_1 + 1 = 6$ and $P(3) = 27a_3 + 9a_2 + 3a_1 + 1 = 15$. This amounts to the matrix equation $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \end{pmatrix}$

In Jupyter Notebook we can solve this system using the `linalg` package in the `numpy` library:

```
import numpy as np
A = np.array([[1,1,1],[2,4,8],[3,9,27]])
b = np.array([[1], [5], [14]])
n = np.linalg.solve(A, b)
n
```

This produces

```
array([[ 0.16666667],
       [ 0.5       ],
       [ 0.33333333]])
```

In other words, $P(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + 1 = \dfrac{n(2n+1)(n+1)}{6} + 1$

(c) Find a good candidate for a closed-form solution.
ANS: See above. But we could have proceeded differently (heh). Compute the general first order difference:

$$\begin{aligned}
\Delta P(m) &= P(m) - P(m-1) \\
&= a_n m^n + a_{m-1} m^{n-1} + \cdots a_1 m + a_0 - (a_n (m-1)^n + a_{m-1}(m-1)^{n-1} + \cdots a_1(m-1) + a_0) \\
&= a_n(m^n - (m-1)^n) + a_{n-1}(m^{n-1} - (m-1)^{n-1} + \cdots + a_1(m - (m-1)) + a_0 - a_0 \\
&= a_n n m^{n-1} + a_{n-1}(n-1)m^{n-2} + \cdots + a_1
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Delta^2 P(m) &= \Delta P(m) - \Delta P(m-1) \\
&= a_n n(n-1)m^{n-2} + a_{n-1}(n-1)(n-2)m^{n-2} + \cdots + a_2
\end{aligned}$$

and continuing in this fashion leads to
$\Delta^n = a_n n! = c \Leftrightarrow a_2 = \dfrac{c}{n!}$

In our case $n = 3$ and $c = 2$ so $a_3 = \dfrac{2}{3!} = \dfrac{1}{3}$
How would you then proceed to find a_{n-1} and so on?

(d) Prove that your candidate solution is the correct closed-form solution.
Proof (by induction) First $P(0) = 1$ checks. Now assume $P(k) = \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k + 1 =$

Then, using the recurrence relation,

$$\begin{aligned}
P(k+1) &= P(k) + (k+1)^2 \\
&= \frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k + 1 + (k^2 + 2k + 1) \\
&= \frac{1}{3}k^3 + \frac{1}{3}(3k^2) + \frac{1}{3}(3k) + \frac{1}{3} + \frac{1}{2}k^2 + k + \frac{1}{2} + \frac{1}{6} + \frac{1}{6}k + 1 \\
&= \frac{1}{3}(k^3 + 3k^2 + 3k + 1) + \frac{1}{2}(k^2 + 2k + 1) + (\frac{1}{6}k + \frac{1}{6}) + 1 \\
&= \frac{1}{3}(k+1)^3 + \frac{1}{2}(k+1)^2 + \frac{1}{6}(k+1) + 1 \\
&= P(k+1)
\end{aligned}$$

There's likely a better way, but that works.

2. (3.2 # 14) Analyze the sequence using sequences of differences. From what degree polynomial does this sequence appear to be drawn? (Don't bother finding the coefficients of the polynomial.)
ANS: Keep computing forward differences until they become constant. To aid in doing this, I wrote a little Python function:

```
c=[1,6,15,100,501,1746,4771,11040,22665,42526,74391]

def forwardDiffs(c):
    k = 1
    x = []
    same = False
    x.append(c)
    print(x[0])
    while not same:
        temp = []
        for i in range(len(x[k-1])-1):
            temp.append(x[k-1][i+1]-x[k-1][i])
        x.append(temp)
        print(x[k])
        same = True
        for j in range(len(x[k])-1):
            same = same and x[k][j+1]==x[k][j]
        k += 1

forwardDiffs(c)
[1, 6, 15, 100, 501, 1746, 4771, 11040, 22665, 42526, 74391]
[5, 9, 85, 401, 1245, 3025, 6269, 11625, 19861, 31865]
[4, 76, 316, 844, 1780, 3244, 5356, 8236, 12004]
[72, 240, 528, 936, 1464, 2112, 2880, 3768]
[168, 288, 408, 528, 648, 768, 888]
[120, 120, 120, 120, 120, 120]
```

y	$\triangle y$	$\triangle^2 y$	$\triangle^3 y$	$\triangle^4 y$	$\triangle^5 y$
1					
6	5				
15	9	4			
100	85	76	72		
501	401	316	240	168	
1746	1245	844	528	288	120
4771	3025	1780	936	408	120
11040	6269	3244	1464	528	120
22665	11625	5356	2112	648	120
42526	19861	8236	2880	768	120
74691	31865	12004	3768	888	120

Evidently, it's a fifth degree polynomial

3. (3.2 # 18) Recall that $n! = 1 \cdot 2 \cdot 3 \cdots (n1) \cdot n$ for $n > 0$, and by definition, $0! = 1$. Prove that

$$F(n) = \begin{cases} 1 & : n = 0 \\ n \cdot F(n-1) & : n > 0 \end{cases}$$

ANS: Proof. (by induction on n.)
Base Case: If $n = 0$, the recurrence relation says that $F(0) = 1$, and the definition says that $0! = 1$, so they match.
Inductive Hypothesis: Suppose as inductive hypothesis that $F(k-1) = (k-1)!$ for some $k > 0$.
Inductive Step: Using the recurrence relation,

$$\begin{aligned} F(k) &= k \cdot F(k-1), \texttt{ by the second part of the recurrence relation} \\ &= k \cdot (k-1)!, \texttt{ by inductive hypothesis} \\ &= k \cdot (1 \cdot 2 \cdot 3 \cdots (k-1)) = k! \end{aligned}$$

so, by induction, $F(n) = n!$ for all $n \geq 0$.

4. (3.2 # 22) Let $f(n) = An^2 + Bn + C$. Show that the expression

$$f(n+1) - f(n)$$

is a linear function of n. (This calculation shows that a quadratic sequence has a linear sequence of differences.)
ANS: $f(n+1) - f(n) = A(n+1)^2 + B(n+1) + C - (An^2 + Bn + C) = 2An + A + B$, which is a linear function of n.

5. (3.2 # 24) Suppose you are given a sequence of numbers $a_1, a_2, a_3, \ldots, a_k$. Explain how to construct a polynomial $p(x)$ such that $p(n) = a_n$ for all $n = 1, 2, 3, \ldots, k$. (Note that this fact, along with Exercise 23, shows that it is possible for a closed-form formula to match a recurrence relation for arbitrarily many terms, without being a valid closed-form solution.)

ANS: Solve the following system of equations for $C_0, C_1, C_2, \ldots, C_{k-1}$.

$$\begin{aligned} a_1 &= C_0 + 1C_1 + 1^2C_2 + 1^3C_3 + \cdots + 1^{k1}C_{k-1} \\ a_2 &= C_0 + 2C_1 + 2^2C_2 + 2^3C_3 + \cdots + 2^{k1}C_{k-1} \\ &\vdots \\ a_k &= C_0 + kC_1 + k^2C_2 + k^3C_3 + \cdots + k^{k1}C_{k-1} \end{aligned}$$

The polynomial $p(n) = C_0 + C_1n + C_2n^2 + \cdots + C_{k-1}n^{k-1}$ will have $p(n) = a_n$ for all $n = 1, 2, 3, \ldots, k$.