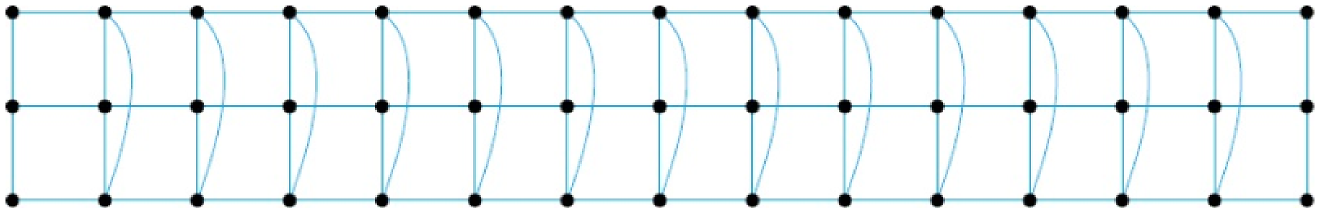
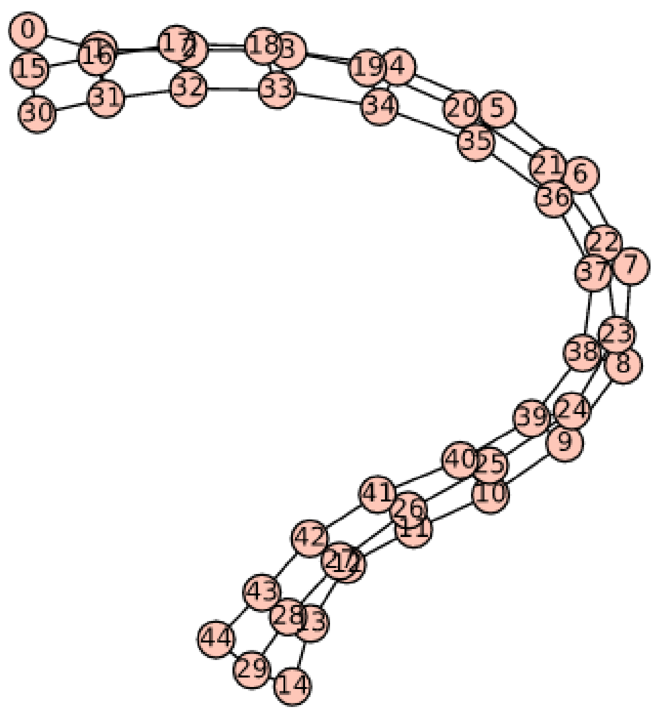
Math 15 - Spring 2017 - Homework 2.6 Solutions

1. (2.6 # 20) The following graph has 45 vertices.



In Sagemath, we can define it like so:

```
dm = {0: [1,15], 1: [2,16,31], 2: [3,17,32], 3: [4,18,33], 4: [5,19,34], \
5: [6,20,35], 6: [7,21,36], 7: [8,22,37], 8: [9,23,38], 9: [10,24,39], \
10: [11,25,40], 11: [12,26,41], 12: [13,27,42], 13: [14,28,43],\
14: [29], 15: [16,30], 16: [17,31], 17: [18,32], 18: [19,33], 19: [20,34],\
20: [21,35], 21: [22,36], 22: [23,37], 23: [24,38], 24: [25, 39], \
25: [26, 40], 26: [27, 41], 27: [28, 42], 28: [29, 43], 29: [44], \
30: [31], 31: [32], 32: [33], 33: [34], 34: [35], 35: [36], 36: [37], \
37: [38], 38: [39], 39: [40], 40: [41], 41: [42], 42: [43], 43: [44]}
DM = Graph(dm); DM
DM.plot()
```

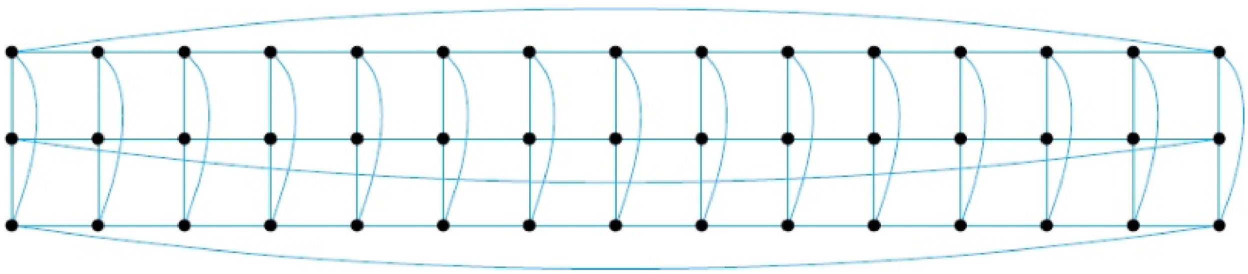


(a) Does this graph have an Euler circuit? Why or why not?
ANS: No, theorem 2.8 states that "If a graph G has an Euler circuit, then all the vertices of G have even degree." This graph has two vertices of odd degree.

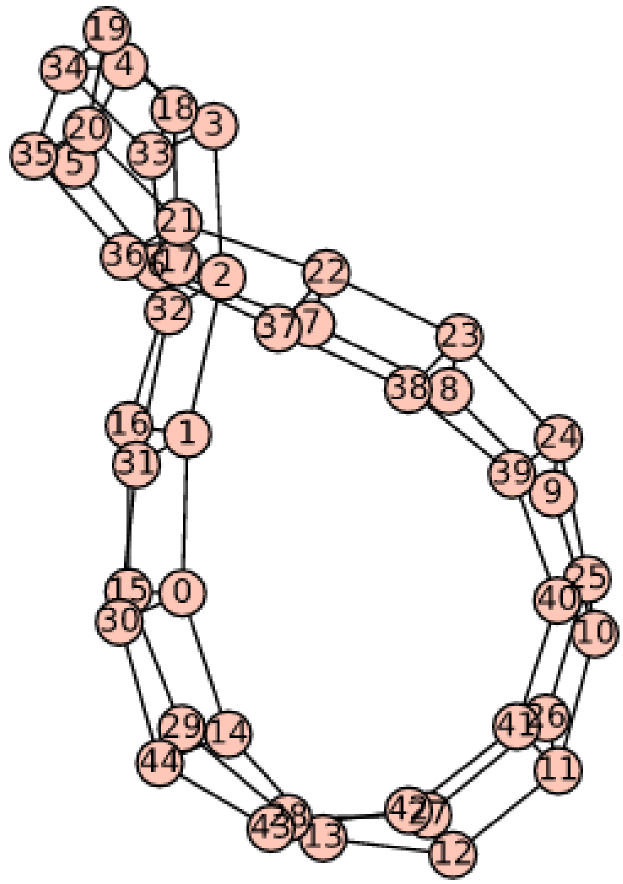
(b) Does this graph have an Euler path? Why or why not?
ANS: Yes, in problem 17 we proved Theorem 2.11:

> If a connected, undirected graph has exactly two vertices v and w of odd degree, then there is an Euler path from v to w.

(c) The graph below is a copy of the above graph, but with some additional edges added so that all of the vertices in the resulting graph have degree four.



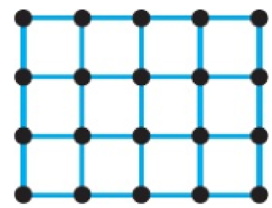
If we add those edges in Sagemath, the graph comes out looking like this:



How many edges does this new graph have? Explain how you can use a theorem from this section to make counting the edges easier.
ANS: The total number of edges is half the sum of all valences (degrees) of the vertices. Since there are 45 vertices, each with degree 4, there are 90 edges.

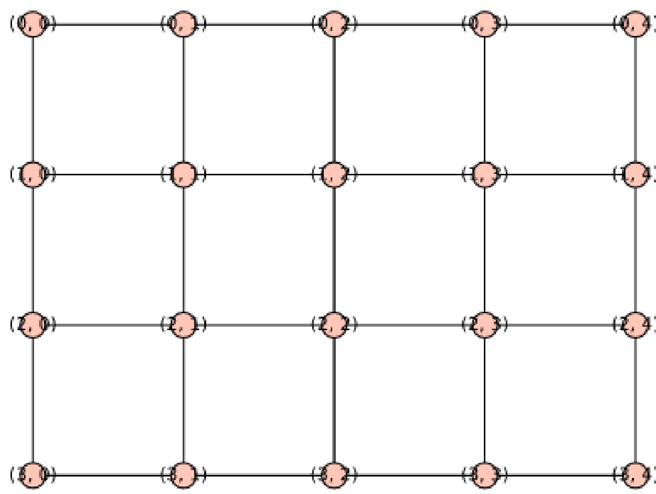
2. (2.6 # 22) For what values of n does K_n have an Euler circuit? Explain.
ANS: In K_n every vertex has degree $n - 1$, so if n is odd, then all vertices have an even degree and by Theorem 2.8, there is an Euler circuit.

3. (2.6 # 24) Find a Hamilton circuit in the following graph.

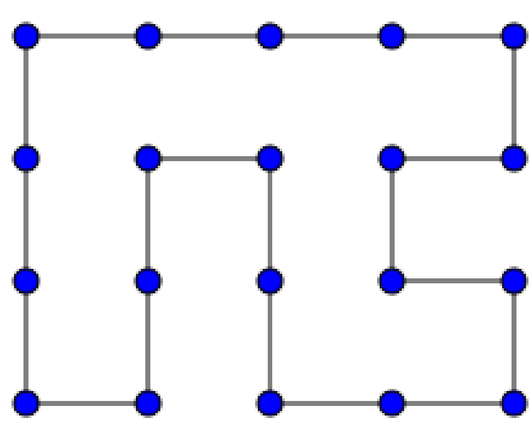


You can go row by row or column by column, or...?
In Sagemath, such a graph is called a gridGraph and can be created like so:

```
g = graphs.Grid2dGraph(4,5)
g.plot()
```



We can specify the Hamilton path like so:
$(0,0),(0,1),(0,2),(0,3),(0,4),(1,4),(1,3),(1,2),(1,1),(1,0),$
$(2,0),(2,1),(2,2),(2,3),(2,4),(3,4),(3,3),(3,2),(3,1),(3.0).$
A Hamilton circuit must end at the starting vertex.
$(0,0),(0,1),(0,2),(0,3),(0,4),(1,4),(1,3),(2,3),(2,4),(3,4),$
$(3,3),(3,2),(2,2),(1,2),(1,1),(2,1),(3,1),(3,0),(2,0),(1,0),(0,0).$



4. (2.6 # 26) Finish the proof of Theorem 2.9

> Let G be an undirected graph, and let $r \in G$. Then G is a tree with root r if and only if G is connected and has no simple circuits.

by proving that, if a graph G has a circuit with some edge e that differs from all the other edges in the circuit, then G has a simple circuit. (Hint: Focus on the sequence of vertices in the given circuit.)
ANS: Let's start by reviewing the partial (mostal) proof presented in Hunter:
($\Rightarrow$) Suppose that G is an undirected tree with root r. Let a, b be two vertices in G. By Definition 2.18, there are paths from r to a and r to b. Therefore, there is a path from a to b via r, so G is connected.

Suppose, to the contrary, that G has a simple circuit

$$v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_0$$

We can relabel this circuit, if necessary, so that $v_0 \neq r$. If $r = v_i$ for some i, then

$$v_0, e_1, \ldots, r \text{ and } r, e_{i+1}, \ldots, v_0$$

are two different simple paths from v_0 to r, contradicting the definition of a tree. If $r \neq v_i$ for any i, then there is a simple path from r to v_0, and combining this path with the sequence

$$v_1, e_1, v_2, \ldots, v_{k-1}, e_k, v_0$$

yields another simple path from r to v_0, a contradiction.
($\Leftarrow$) Now suppose that G is a connected undirected graph with no simple circuits. Let $v \neq r$ be a vertex in G. Since G is connected, there is a path from r to v. If this path does not repeat any vertices, then it does not repeat any edges, so it is simple. If this path has the form

$$r, e_1, \ldots, e_i, a, e_{i+1}, \ldots, e_k, a, e_{k+1}, \ldots, e_n, v$$

for some vertex a, then we can replace it with a shorter path

$$r, e_1, \ldots, e_i, a, e_{k+1}, \ldots, e_n, v$$

that still goes from r to v. Furthermore, we can repeat this shortening procedure until there are no repeated vertices. So there is a simple path from r to v. To show that G is a tree with root r, we need to show that this path is unique. But if there were two distinct simple paths

$$r, e_1, v_1 \ldots, e_n, v \text{ and } r, d_1, w_1 \ldots, d_n, v$$

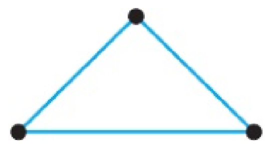
from r to v, we could combine them to form a circuit

$$r, e_1, v_1 \ldots, e_n, v, d_m, w_{m-1}, \ldots, w_1, d_1, r$$

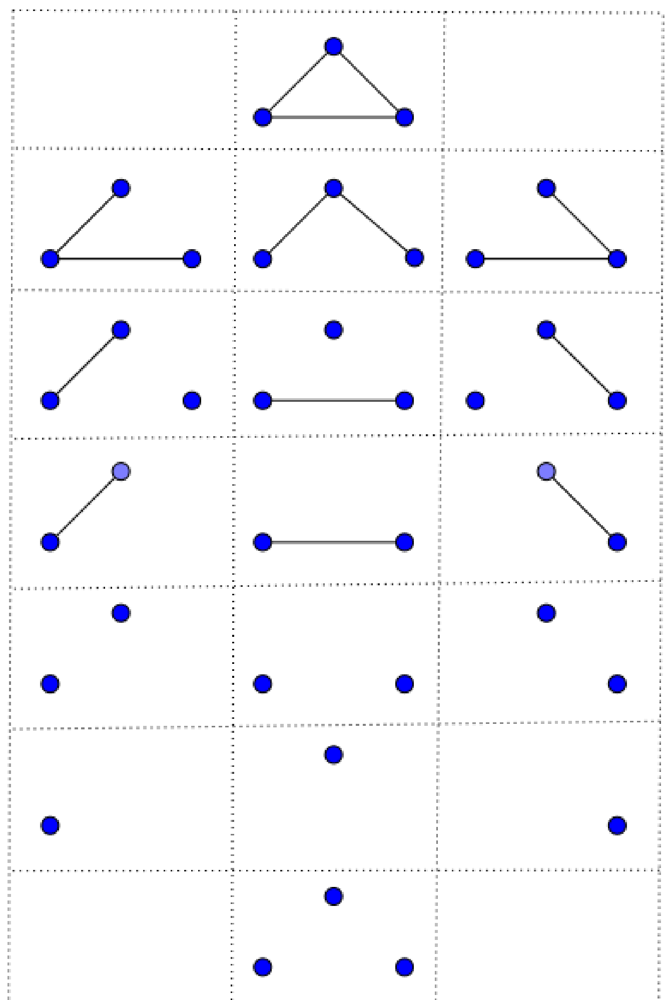
in G. While this circuit may not be simple, it must contain a simple circuit, since there is at least one d_i that differs from all the e_j's. (This detail is left as Exercise 26.) This contradicts that G has no simple circuits.
Claim: Any circuit $r, e_1, v_1 \ldots, e_n, v, d_m, w_{m-1}, \ldots, w_1, d_1, r$ must contain a simple circuit.
Proof: This amounts to showing there is at least one d_i that differs from all the e_j's, since we can simply remove any repeated edges and the vertices in between and still have a circuit. We repeatedly remove repeated edges until we have no more and the resulting circuit is simple. More formally, suppose that e is an edge that is used only once in the circuit and e connects x to y so that the circuit includes the path $x, e_1, \ldots, e_i, y$. If all those vertices are different, then so are the edges, and we are done. If a vertex

is repeated, then snip out the sub-loop that that repeated vertex forms. Continue doing this until all the sub-loops are removed. What's left is a simple circuit.

5. (2.6 # 28) Let G be a graph with vertex set V_G and edge set E_G. A subgraph of G is a graph with vertex set $V_H \subseteq V_G$ and edge set $E_H \subseteq E_G$. Write down all the subgraphs of the following graph.



ANS:



6. (2.6 # 30) Let G be a connected, undirected graph. Prove that there is a subgraph T of G such that T contains all the vertices of G, and T is a tree. (Such a subgraph is called a spanning tree.) Give a constructive proof that explains how to construct a spanning tree of a graph.
ANS: If G has no circuits, then G is a tree (Theorem 2.9), and we are done. If G has a circuit, remove one of the edges of this circuit from G to form subgraph H that is still connected and includes all the vertices of G. Keep doing this until all the circuits are removed. The resulting subgraph T will contain all the vertices of G and will have no simple circuits. By Theorem 2.9, it is a tree.