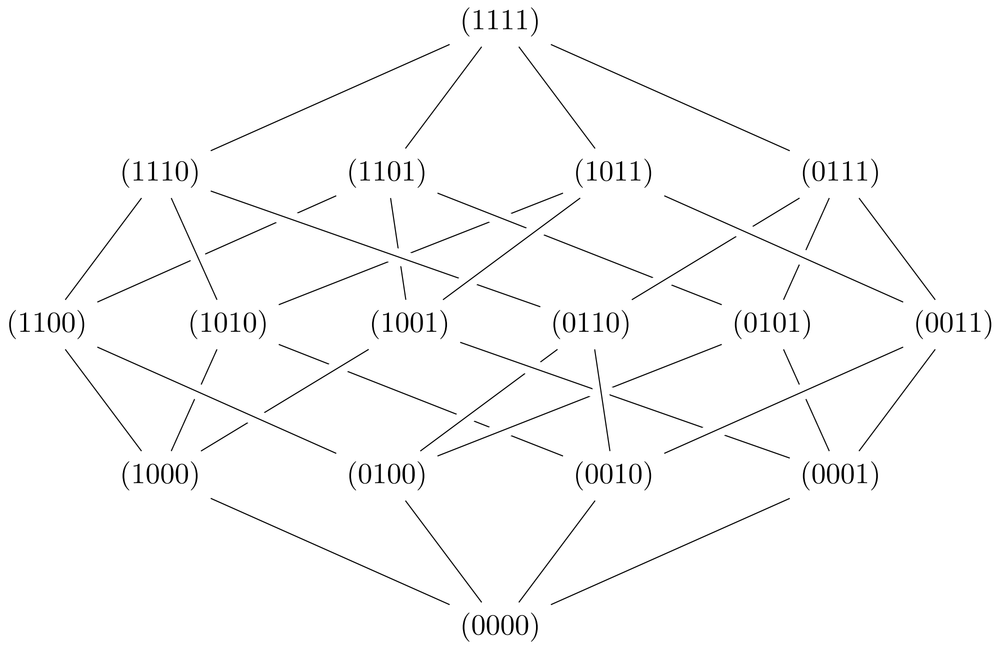
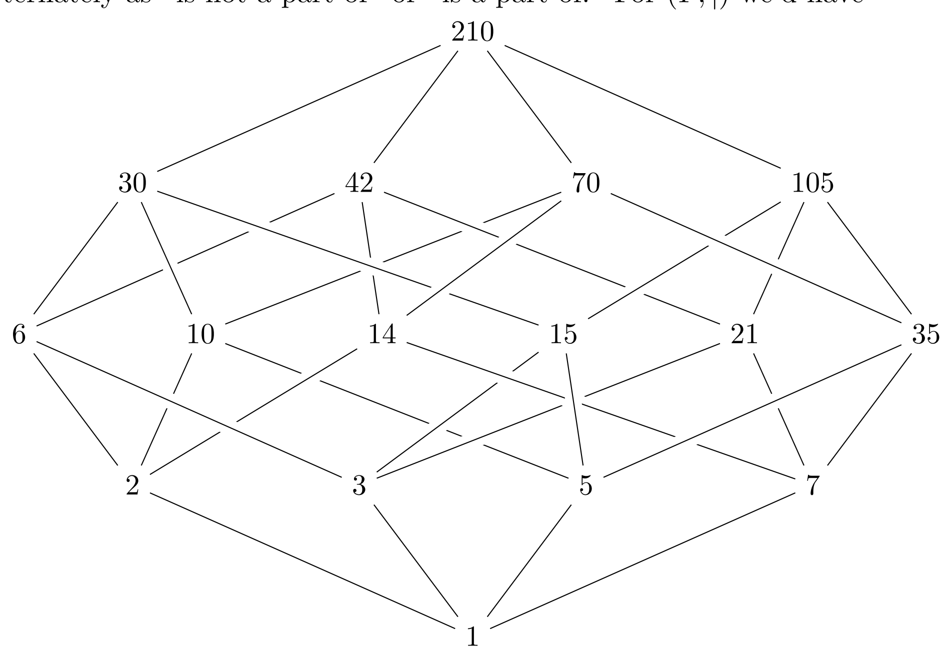
Math 15 - Spring 2017 - Homework 2.5 Solutions

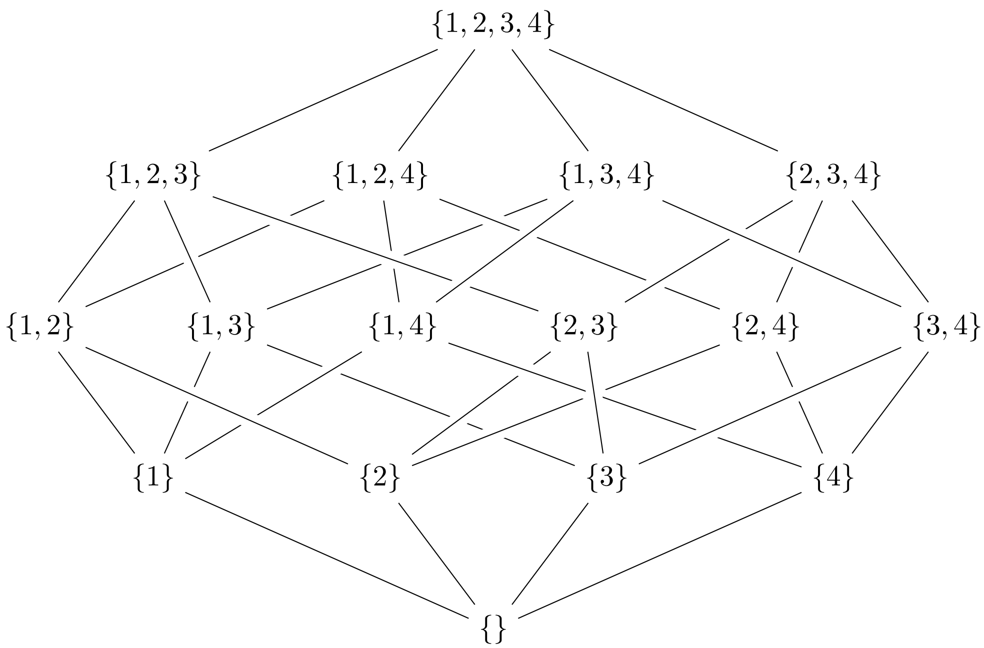
1. (2.5 # 20) Let $F \subseteq N$ be the set of all factors of 210. Draw the Hasse diagrams for $(F, |)$ and $(P(1,2,3,4), \subseteq)$ in a way that shows these two posets are isomorphic.
   ANS: $210 = 2 \cdot 3 \cdot 5 \cdot 7$ so the poset's Hass diagram will be isomorphic to the poset of $(P(1,2,3,4), \subseteq)$ as shown below:



   This is an all-purpose isomorphic version of the Hasse diagram where "0" and "1" are interpretted alternately as "is not a part of" or "is a part of." For $(F, |)$ we'd have



   or with $(P(1,2,3,4), \subseteq)$ we'd have



2. (2.5 # 22) Let B be the set of all four-digit binary strings; that is,

$$B = \{0000, 0001, 0010, 0011, \ldots, 1110, 1111\}$$

Define a relation $\triangleleft$ on B as follows: Let $x, y \in B$, where $x = x_1x_2x_3x_4$ and $y = y_1y_2y_3y_4$. We say that $x \triangleleft y$ if $x_i \leq y_i$ for $i = 1, 2, 3, 4$. In other words, $x \triangleleft y$ if y has a 1 in every position where x does. So, for example, $0101 \triangleleft 0111$ and $0000 \triangleleft 0011$, but $1010 \ntriangleleft 0111$. The relation $\triangleleft$ is called the bitwise $\leq$. Show that $(B, \triangleleft)$ is a poset.
ANS: To show show $(B, \triangleleft)$ is a poset it is necessary to show it satisfies all three of the following properties.

   1. *Reflexivity.* $\forall a \in S, a\ R\ a$.
      The "or equal" part takes care of reflexivity.
   2. Transitivity. $\forall a, b, c \in S$, if $a\ R\ b$ and $b\ R\ c$, then $a\ R\ c$.
      ANS: This follows directly from the transitivity of $\leq$.
   3. Antisymmetry. $\forall a, b \in S$, if $a\ R\ b$ and $b\ R\ a$, then $a = b$.
      ANS: Again this is an immediate consequence of the fact that $\leq$ is antisymmetric.

3. (2.5 # 24) In $(B, \triangleleft)$, give a counterexample to show that

$$0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1001, 1000, 1010, 0011, 1100, 1101, 1110, 1111$$

is not a valid topological sort of the elements of B.
ANS: In the transition from 0011 to 0100, for example, we have $0011 \ntriangleleft 0100$.

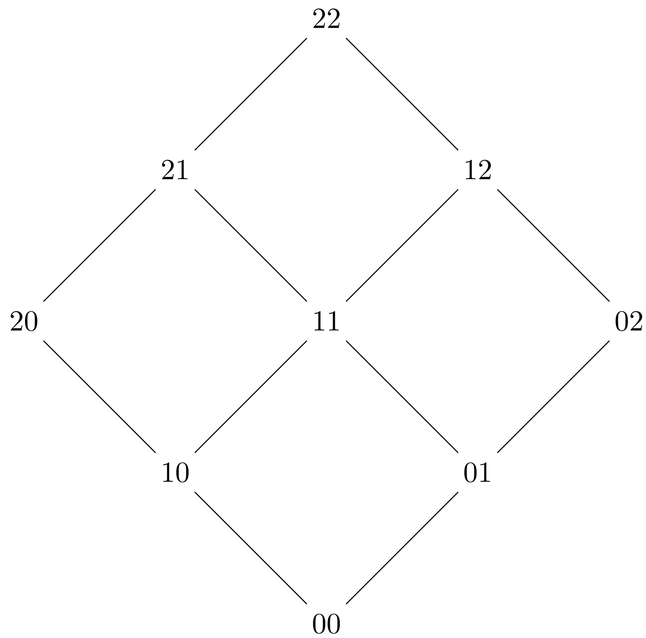
4. (2.5 # 28) Let $m, n \in \mathbb{Z}$ and suppose $m \leq n$. In the poset $(\mathbb{Z}, \leq)$, what are $m \wedge n$ and $m \vee n$?
ANS: Think of $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. Then $m \leq n \Rightarrow m \wedge n = m$ and $m \vee n = n$.

5. (2.5 # 30) Let T be the set of all two-digit ternary strings; that is,

$$T = \{00, 01, 02, 10, 11, 12, 20, 21, 22\}$$

and consider the poset $(T, \triangleleft)$ where $\triangleleft$ is the bitwise $\leq$. This poset is, in fact, a lattice; that is, every pair of elements $a, b \in T$ has both a meet and a join and the properties of commutativity, associativity and absorption are satisfied. To be sure, it is helpful to draw the Hasse diagrams suggested in exercise

29:



(a) Theorem 2.4: Let X be a finite set. Suppose that the poset $(X, \preceq)$ is a Boolean algebra. Then $|X| = 2^n$ for some $n \in \mathbb{N}$. This theorem can be used to show that $(T, \lhd)$ is not a Boolean algebra. How?
ANS: Pretty straight-forward, $|T| = 9$ is not a power of 2.

(b) Find an element of $(T, \lhd)$ that has no complement. Explain why.
ANS: Recall that **complements** are required for a poset to be a Boolean algebra, and the complements requirement is that, $\forall x \in X$, $\exists \neg x \in X$ such that $x \wedge \neg x = 0$ and $x \vee \neg x = 1$. However, 11 has no complement. To see this, note that for all $x \in X$, $x \vee 11 \geq 11$, but that $11 \vee 20 = 21$ and $11 \vee 02 = 12$, so $\neg 11$ must be 22. However, $11 \wedge 22 = 11 \neq 0$, so $22 \neq \neg 11$, so the $\neg 11$ does not exist.