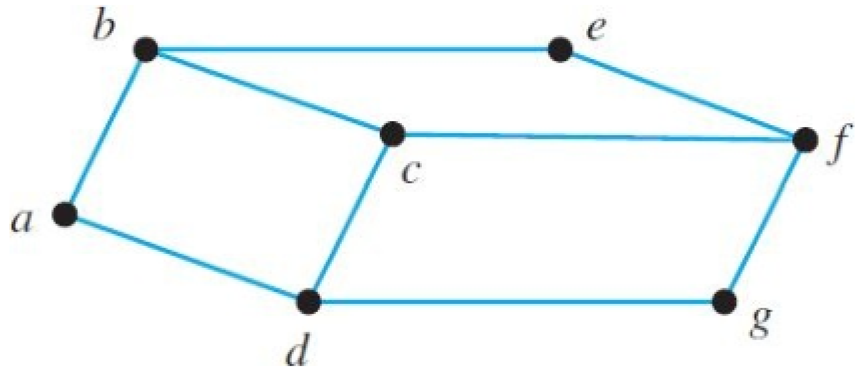
Math 15 - Spring 2017 - Homework 2.4 Solutions

1. (2.4 # 25 (not assigned)) Let G be a connected, undirected graph, and let V be the set of all vertices in G. Define a relation R on V as follows: for any vertices $a, b \in V$, aRb if there is a path from a to b with an even number of edges. (A path may use the same edge more than once.) Prove that R is an equivalence relation.
   ANS: We need to show reflexivity (just go back and for from a to a neighbor and you have a path of length 2), symmetry (the graph is undirected so $aRb \Leftrightarrow bRa$) and transitivity (the sum of any two even numbers is even.)

2. (2.4 # 26) Suppose the equivalence relation of Exercise 25 is defined on the vertices of the following graph. What are the equivalence classes?



   ANS: The equivalence classes are $\{b, d, f\}$ and $\{a, c, g, e\}$.

3. (2.4 # 28) Lemma 2.1 states that the " $\equiv$ `mod` n'' relation on $\mathbb{Z}$ is transitive. Show that it is also symmetric and reflexive.
   ANS: For symmetry, let $a \equiv b \mod n$. Then $a = b + kn$ for some $k \in \mathbb{Z}$, so $b = a - kn$, hence $b \equiv a \mod n$.
   For reflexivity just note that $\forall a \in \mathbb{Z}$, $a = a + 0 \cdot n$, so $a \equiv a \mod n$.

4. (2.4 # 32) Construct the addition and multiplication tables for $\mathbb{Z}/4$.

ANS:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

5. (2.4 # 34) Calculate the tenth digit of the ISBN whose first nine digits are 039481500
   ANS: We need $(1\cdot 0+2\cdot 3+3\cdot 9+4\cdot 4+5\cdot 8+6\cdot 1+7\cdot 5+8\cdot 0+9\cdot 0+10\cdot a_{10}) \mod 11 = (130+10a_{10}) \mod 11 = 9 + 10a_{10} = 11k$ so choose $a_{10} = 9$.

6. (2.4 # 36) Is 0060324814 a valid ISBN number?
   ANS: $(3\cdot 6 + 5\cdot 3 + 6\cdot 2 + 7\cdot 4 + 8\cdot 8 + 9\cdot 1 + 10\cdot 4) \mod 11 = 186 \mod 11 \neq 0$, so ...no.

7. (2.4 # 38) Show that the check digit will always detect the error of changing a single digit. Hint: The proof has something to do with the multiplication table for $\mathbb{Z}/11$.

   Suppose digit k is changed from a_k to $b_k \neq a_k$. The error will be detected if

$$(1\cdot a_1 + \cdots + k\cdot a_k + (k+1)\cdot a_{k+1} + \cdots + 9a_9)(1\cdot a_1 + \cdots + k\cdot b_k + (k+1)\cdot a_{k+1} + \cdots + 9a_9$$

   is not a multiple of 11 (not $\equiv 0 \mod 11$). But the difference is $k\cdot(a_k - b_k)$, the product of two nonzero (mod 11) integers is never 0.