Math 15 - Spring 2017 - Homework 1.5 Solutions

1. (1.5 # 20)Prove: Let x and y be real numbers. If x is rational and y is irrational, then $x+y$ is irrational.
   Proof (by contradiction): Suppose, to the contrary that $x + y = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$. Then $y = \frac{a}{b} - x$, but x is rational so we can write it as a ratio of integers and substitute to get $y = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$ and since $\mathbb{Z}$ is closed under the operations of addition and multiplication, we have written y as a ratio of integers, contradicting the assumption that y is irrational. $\rightarrow\leftarrow$.

2. (1.5 # 22) Recall the Badda-Bing axiomatic system of Example 1.17. Prove:

   If q and r are distinct bings, both of which are hit by baddas x and y, then $x = y$.

   Recall the axioms:

   1. Every badda hits exactly four bings.
   2. Every bing is hit by exactly two baddas.
   3. If x and y are distinct baddas, each hitting bing q, then there are no other bings hit by both x and y.
   4. There is at least one bing.

   Proof (by contradiction): Suppose, to the contrary, that q and r are distinct bings (points), both of which are hit by baddas (lines) x and y, and that $x \neq y$. This contradicts axiom 3 since x and y are now distinct baddas each hitting bing q and therefore cannot also hit bing r $\rightarrow\leftarrow$.

3. (1.5 # 23 (not assigned)) Two common axioms for geometry are as follows. The undefined terms are "point," "line," and "is on."

   1. For every pair of points x and y, there is a unique line such that x is on l and y is on l.
   2. Given a line l and a point x that is not on l, there is a unique line m such that x is on m and no point on l is also on m.

   Recall that two lines l and m are parallel if there is no point on both l and m. In this case we write $l \| m$. Use this definition along with the above two axioms to prove the following.
   Let l, m, and n be distinct lines. If $l \| m$ and $m \| n$, then $l \| n$.

   Proof (by contradiction): Suppose, to the contrary, that l, m, and n are lines with $l \| m$ and $m \| n$ but l is not parallel to n. By definition, this means that l and n intersect at some point x on both l and n. Now since $l \| m$, x cannot be on m. Axiom 2 then tells us that there is a **unique** line through x that is parallel to m. This contradicts the assumption that l and n are distinct lines through x, each parallel to m $\rightarrow\leftarrow$.

4. (1.5 # 24) Recall Example 1.16. In the axiomatic system for four-point geometry, prove the following assertion using a proof by contradiction:

   Suppose that a and b are distinct points on line u. Let v be a line such that $u \neq v$. Then a is not on v or b is not on v.

   It's helpful to have the axioms handy:

   Undefined terms: point, line, is on
   Axioms:
   1. For every pair of distinct points x and y, there is a unique line l such that x is on l and y is on l.
   2. Given a line l and a point x that is not on l, there is a unique line m such that x is on m

and no point on l is also on m.

3. There are exactly four points.
4. It is impossible for three points to be on the same line.

Proof (by contradiction): Suppose, to the contrary, x and y are distinct points on line u, and let v be a different line ($u \neq v$) that contains point x. Since $u \neq v$, this contradicts Axiom 1.