Math 15 - Spring 2017 - Homework 1.3 and 1.4 Solutions

1. (1.3#12) Write the following statements in predicate logic. Define what your predicates are. Use the domain of all quadrilaterals.
   (a) All rhombuses are parallelograms.
   ANS: $\forall x(R(x) \to P(x))$
   (b) Some parallelograms are not rhombuses.
   $\exists x(P(x) \wedge \neg R(x))$

2. (1.3#14) In the domain of all people, consider the following predicate.

$$P(x, y) = \text{“x needs to love y”}$$

   (a) Write the statement “Everybody needs somebody to love” in predicate logic.
   ANS: $\forall x \exists y L(x, y)$
   (b) Formally negate your statement from part (a).
   $\neg(\forall x \exists y L(x, y)) = \exists x \forall y \neg L(x, y)$
   (c) Write the English translation of your negated statement. There’s a person who doesn’t need love from anyone.

3. (1.3#16) Any equation or inequality with variables in it is a predicate in the domain of real numbers. For each of the following statements, tell whether the statement is true or false.
   (a) $(\forall x)(x^2 > x)$
   ANS: False. A counterexample is $x = \frac{1}{2}$
   (b) $(\exists x)(x^2 - 2 = 1)$
   ANS: Surely: $x = \pm\sqrt{3}$
   (c) $(\exists x)(x^2 + 2 = 1)$
   Nope: The square of any real number must be positive. I’m certain.
   (d) $(\forall x)(\exists y)(x^2 - y = 4)$
   ANS: Yes, $y = 4 - x^2 \in \mathbb{R}$
   (e) $(\exists y)(\forall x)(x^2 + y = 4)$
   False. That would mean that for some y the equation is an identity.

4. (1.3#18)(a) Write the following statement in predicate logic, and negate it. Say what your predicates are, along with the domains.
   Let x and y be real numbers. If x is rational and y is irrational, then $x + y$ is irrational.
   ANS: Let $R(x)$ be the statement that x “is rational”, in the universe of real numbers. The statement then translates like so:

$$(\forall x)(\forall y)((R(x) \wedge \neg R(y)) \to \neg R(x + y))$$

5. (1.3#20) Let the following predicates be given in the domain of all triangles.

   $R(x) =$“x is a right triangle.”
   $B(x) =$“x has an obtuse angle.”

   Consider the following statements:

   $S_1 = \neg(\exists x)(R(x) \wedge B(x))$
   $S_2 = (\forall x)(R(x) \to \neg B(x))$

   (a) Write a proof sequence to show that $S_1 \Leftrightarrow S_2$

$$\begin{aligned}
S_1 &= \neg(\exists x)(R(x) \wedge B(x)) \\
&\Leftrightarrow (\forall x)\neg(R(x) \wedge B(x)) \text{ , exist. neg.} \\
&\Leftrightarrow (\forall x)(\neg R(x) \vee \neg B(x)) \text{ , De Morgan} \\
&\Leftrightarrow (\forall x)(R(x) \rightarrow \neg B(x)) \text{ , implication} \\
&= S_2
\end{aligned}$$

(b) Write S_1 in ordinary English.
ANS: There is no triangle that is both right and has obtuse angles.

(c) Write S_2 in ordinary English.
ANS: All right triangles have no obtuse angles.

6. (1.3#22) Let the following predicates be given. The domain is all cars.

$F(x) =$"x is fast."
$S(x) =$"x is a sports car."
$E(x) =$"x is expensive."
$A(x, y) =$"x is safer than y."

(a) Write the following statements in predicate logic.
   (a) All sports cars are fast. ANS: $(\forall x)(S(x) \rightarrow F(x))$
   (b) There are fast cars that aren't sports cars. ANS: $(\exists x)(F(x) \wedge \neg S(x))$
   (c) Every fast sports car is expensive. ANS: $(\forall x)((F(x) \wedge S(x)) \rightarrow E(x))$

(d) Write the following predicate logic statement in everyday English. Don't just give a word-for-word translation; your sentence should make sense.

$$(\forall x)[S(x) \rightarrow (\exists y)(E(y) \wedge A(y, x))]$$

ANS: For every sports car, there is some expensive car that is safer.

(e) Formally negate the statement from part (b). Simplify your negation so that no quantifier or connective lies within the scope of a negation. State which derivation rules you are using.

$$\begin{aligned}
&\neg(\forall x)[S(x) \rightarrow (\exists y)(E(y) \wedge A(y, x))] \\
&\Leftrightarrow (\exists x)\neg[S(x) \rightarrow (\exists y)(E(y) \wedge A(y, x))] \text{ , univ. neg.} \\
&\Leftrightarrow (\exists x)\neg[\neg S(x) \vee (\exists y)(E(y) \wedge A(y, x)))] \text{ , implication} \\
&\Leftrightarrow (\exists x)[\neg\neg S(x) \wedge \neg(\exists y)(E(y) \wedge A(y, x)))] \text{ , De Morgan} \\
&\Leftrightarrow (\exists x)[S(x) \wedge \neg(\exists y)(E(y) \wedge A(y, x)))] \text{ , double neg.} \\
&\Leftrightarrow (\exists x)[S(x) \wedge (\forall y)\neg(E(y) \wedge A(y, x)))] \text{ , exist. neg.} \\
&\Leftrightarrow (\exists x)[S(x) \wedge (\forall y)(\neg E(y) \vee \neg A(y, x)))] \text{ , De Morgan} \\
&\Leftrightarrow (\exists x)[S(x) \wedge (\forall y)(E(y) \rightarrow \neg A(y, x)))] \text{ , implication}
\end{aligned}$$

(f) Give a translation of your negated statement in everyday English.
ANS: There is a sports car that is at least as safe as every expensive car.

7. (1.3#24)

(a) Give an example of a pair of predicates $P(x)$ and $Q(x)$ in some domain to show that the $\exists$ quantifier does not distribute over the $\wedge$ connective. THat is, give an example to show that the statements

$$(\exists x)(P(x) \wedge Q(x) \text{ and } (\exists x)P(x) \wedge (\exists x)Q(x)$$

are not logically equivalent.
ANS: In the domain of real numbers, let $G(x)$ denote "$x > 0$" and let $L(x)$ denote "$x < 0$." The statement that $(\exists x)(G(x) \wedge L(x))$ is false: no number can be both positive and negative. However, the statement that $(\exists x)G(x) \wedge (\exists x)L(x)$ is true: there are some numbers that are positive, and some that are negative.

(b) It is true, however, that $\exists$ distributes over $\vee$. That is,

$$(\exists x)P(x) \vee Q(x)) \Leftrightarrow (\exists x)(P(x) \vee (\exists x)Q(x)$$

is an equivalence rule for predicate logic. Verify that your example from part (a) satisfies this equivalence.
ANS: Saying that there is a number that is positive or negative is the same as saying that there is a positive number or there is a negative number.

8. (1.4#22) In four-point geometry, do triangles exist? In other words, is it possible to have three distinct points, not on the same line, such that a line passes through each pair of points? Why or why not?
ANS: Yes. If we define a triangle as three points and the lines that connect any three pairs of points. THen, since there are four points (Axiom 3), and three can't be on the same line (Axiom 4). Every pair of distinct points determines a line (Axiom 1), so any three points and the three lines they determine will form a triangle.

9. (1.4#24) Consider the following model for four-point geometry.
   Points: 1,2,3 4
   Lines $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$
   A point "is on" a line if the line's pair contains the point.
   (a) Give a pair of parallel lines in this model. (Refer to Definition 1.8.) ANS: $(2, 3)$ and $(1, 4)$ are parallel–they have no common point.
   (b) Give a pair of intersecting lines in this model. (Use your definition from Exercise ANS: $(2, 3)$ and $(2, 4)$ intersect at 2.

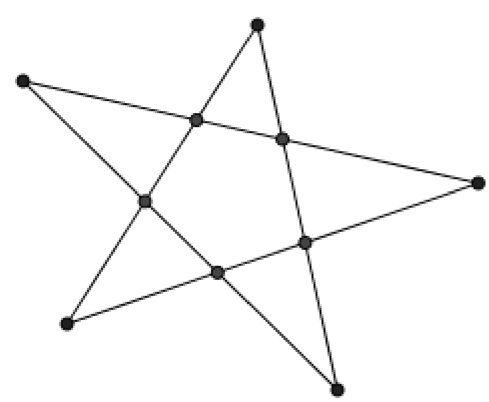
10. (1.4#26) Consider the following definition in the system of Example 1.17.

   Definition. Let x and y be distinct baddas. We say that a bing q is a boom of x and y, if x hits q and y hits q.

   Rewrite Axiom 3 using this definition.
   ANS: Two distinct baddas have at most one boom.

11. (1.4#28) Refer to Example 1.17 and Figure 1.4. Describe a different model, using squares and vertices, where all the squares are the same size.
   ANS: An infinite checkerboard or a finite checkerboard wrapped around the surface of a donut.

12. (1.4#30) Describe a model for Example 1.17 with 10 bings, where a "badda" is a line segment and a "bing" is a point.



ANS:

Also: