Math 15 - Spring 2017 - Homework 1.1 and 1.2 Solutions

1. (1.1#1) Let the following statements be given.

$p =$ "There is water in the cylinders."
$q =$ "The head gasket is blown."
$r =$ "The car will start."

   (a) Translate the following statement into symbols of formal logic.
   If the head gasket is blown and there's water in the cylinders, the the car won't start.
   ANS: $q \wedge p \to \neg r$

   (b) Translate the following formal statement into everyday English.
   $r \to \neg(q \vee p)$
   ANS: If the car will start then neither is the head gasket blown, nor is there water in the cylinders.

2. (1.1#2) Let the following statements be given.

$p =$ "You are in Seoul."
$q =$ "You are in Kwangju."
$r =$ "You are in South Korea."

   (a) Translate the following statement into symbols of formal logic.
   If you are not in South Korea, then you are not in Seoul or Kwangju.
   ANS: $\neg r \to \neg(p \vee q)$

   (b) Translate the following formal statement into everyday English.
   $q \to (r \wedge \neg p)$
   ANS: If you are in Kwangju then you are in South Korea but not in Seoul."

3. (1.1#4)Let s be the following statement.
   If you are studying hard, then you are staying up late at night.

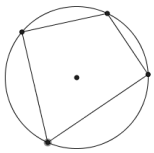
   (a) Give the converse of s.
   If you are staying up late at night, then you are studying hard.

   (b) Give the contrapositive of s.
   ANS: If you are not staying up late at night, then you are not studying hard.

4. (1.1#6) Give an example of a quadrilateral that shows that the *converse* of the statement is false.

   If a quadrilateral has a pair of parallel sides, then it has a pair of supplementary angles.

   Any circular quadrilateral will have two pairs of supplementary angles (opposite angles are supplementary, but only the square or trapezoid will have parallel sides.



5. (1.1#8) Give an example of a true if-then statement whose converse is also true.
   ANS: , "If $3x = 12$, then $x = 4$."

6. (1.1#10) Use truth tables to establish the following equivalencies.

   (a) Show that $\neg(p \vee q)$ is logically equivalent to $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

   (b) Show that $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

   These equivalencies are known as *De Morgan's laws*, after the nineteenth century logician Augustus De Morgan.

7. (1.1#12) Use truth tables to show that $(a \vee b) \wedge (\neg(a \wedge b))$ is logically equivalent to $a \leftrightarrow \neg b$. (This arrangement of T/F values is sometiemes called the *exclusive or* of a and b.)

a	b	$\neg b$	$a \vee b$	$a \wedge b$	$\neg(a \wedge b)$	$(a \vee b) \wedge (\neg(a \wedge b)$	$a \leftrightarrow \neg b$
T	T	F	T	T	F	F	F
T	F	T	T	F	T	T	T
F	T	F	T	F	T	T	T
F	F	T	F	F	T	F	F

8. (1.1#14) Let the following statements be given.

$p =$"Andy is hungry."
$q =$"The refrigerator is empty."
$r =$"Andy is mad."

(a) Use connectives to translate the following statement into formal logic.

If Andy is is hungry and the refrigeartor is empty, then Andy is mad.

ANS: $(p \wedge q) \rightarrow r$

(b) Construct a truth table for the statement.

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

(c) Suppose that the statement given in part (a) is true, and suppose also that Andy is not mad and the refrigerator is empty. Is Andy hungry? Explain how to justify your answer using the truth table.

ANS: The sixth row of the truth table is the only row in which $(p \wedge q) \rightarrow r$ is T, r is F, and q is T. Therefore p is F, that is, Andy is not hungry.

9. (1.1#16) Use truth tables to prove the following *distributive properties* for propositional logic.

(a) $p \wedge (q]veer)$ is logically equivalent to $(p \wedge q) \vee (p \wedge r)$

p	q	r	$P \wedge q$	$p \wedge r$	$q \vee r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	T	F	F
F	T	F	F	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

(b) $p \vee (q \wedge r)$ is logically equivalent to $(p \vee q) \wedge (p \vee r)$

p	q	r	$P \vee q$	$p \vee r$	$q \wedge r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

10. (1.1#18) Mathematicians say that "statement P is *stronger* than statement Q" if Q is true whenever P is true, but not conversely. (in other words, "P is stringer that Q" means that $P \to Q$ is always true, but $Q \to P$ is not true, in general.) Use truth tables to show the following.

    (a) $a \wedge b$ is stronger than a.

a	b	$a \wedge b$	$(a \wedge b) \to a$	$a \to (a \wedge b)$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	T	T

    (b) a is stronger than $a \vee b$

a	b	$a \vee b$	$a \to (a \vee b)$	$(a \vee b) \to a$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	F
F	F	F	T	T

    (c) $a \wedge b$ is stronger than $a \vee b$

a	b	$a \wedge b$	$a \vee b$	$(a \wedge b) \to (a \vee b)$	$(a \vee b) \to (a \wedge b)$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	F	T	T	F
F	F	T	T	F	T

    (d) b is stronger that $a \to b$.

a	b	$a \to b$	$b \to (a \to b)$	$(a \to b) \to b$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	F
F	F	T	T	F

11. (1.1#22) Mathematicians say that "Statement P is a *sufficient condition* for statement Q" if $P \Rightarrow Q$ is true. In other words, in order to know that Q is true, it is sufficient to know that P is true. Let x be an integer. Give a sufficient condition on x for $x/2$ to be an integer.
    SOLN: For $x/2$ to be an even integer, it is sufficient that x be divisible by 100.

12. (1.1#24) Let Q be a quadrilateral. Give a sufficient but not necessary condition for Q to be a parallelogram.
    All four sides of Q are equal.

13. (1.1#26) Often a complicated expression in formal logic can be simplified. For example, consider the statement $S = (p \wedge q) \vee (p \wedge \neg q)$.

    (a) Construct a truth table for S.

p	q	$p \wedge q$	$\neg q$	$p \wedge \neg q$	S
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	F	F	F
F	F	F	T	F	F

    (b) Find a simpler expression that is logically equivalent to S.
    ANS: The statement p is logically equivalent to S.

14. (1.1#28) The NAND connective $\uparrow$ is defined by the following truth table.

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Use truth tables to show that $p \uparrow q$ is logically equivalent to $\neg(p \wedge q)$. (This explains the name NAND: Not AND.) ANS: This is pretty obvious!

15. (1.1#30) Write $\neg p$ in terms of p and $\uparrow$
ANS: $p \uparrow p$

16. (1.1#32) Use symbols of propositional logic to explain the difference between the following two statements.

$p =$"You are in Seoul."
$q =$"My team will win if I yell at the TV."
$r =$"My team will win only if I yell at the TV."

ANS: Let w be the statement "my team will win" and let y be the statement "I yell at the TV." The first statement ("my team will win if I yell at the TV") is $y \to w$, and the second statement is $w \to y$. The applicable meaning of "only" is "exclusively; solely."

17. (1.2#2) Fill in the reasons in the following proof sequence. Make sure you indicate which step(s) each derivation rule refers to.

Statements	Reasons
1. $q \wedge r$	given
2. $\neg(\neg p \wedge q)$	given
3. $\neg\neg p \vee \neg q$	De Morgan, 2
4. $p \vee \neg q$	Double negation, 3
5. $\neg q \vee p$	commutativity, 4
6. $q \to p$	implication, 5
7. q	simplification, 1
8. p	*modus ponens*, 7,6

18. (1.2#6) Fill in the reasons in the following proof sequence. Make sure you indicate which step(s) each derivation rule refers to.

Statements	Reasons
1. $p \wedge (q \vee r)$	given
2. $\neg(p \wedge q)$	given
3. $\neg p \vee \neg q$	De Morgan, 2
4. $\neg q \vee \neg p$	commutativity, 3
5. $q \to \neg p$	implication, 4
6. p	simplification, 1
7. $\neg\neg p$	double negation, 6
8. $\neg q$	*modus tollens*, 7,5
9. $(q \vee r) \wedge p$	commutativity, 1
10. $q \vee r$	simplification, 9
11. $r \vee q$	commutativity, 10
12. $\neg(\neg r) \vee q$	double negation, 11
13. $\neg r \to q$	implication, 12
14. $\neg(\neg r)$	*modus tollens*, 8, 13
15. r	double negation, 14
16. $p \wedge r$	conjuc5tion, 6, 15

19. (1.2#8) Which derivation rule justifies the following argument?

    If n is a multiple of 4, then n is even. However, n is not even. Therefore, n is not a multiple of 4.
    ANS: *modus tollens*

20. (1.2#10) Let Q be a quadrilateral. Given the statements

    If Q is a rhombus, then Q is a parallelogram.
    Q is not a parallelogram.

    what statement follows by *modul tollens*?
    ANS: Q is not a rhombus.

21. (1.2#12) Write a statement that follows from the statement

    It is sunny and warm today.

    by the simplification rule.
    ANS: It is sunny today.

22. (1.2#14) Recall Exercise 31 of Section 1.1. Suppose that all the following status reports are correct:

    - Processor B is not working and processor C is working.
    - Processor A is working if and only if processor B is working.
    - At least one of the two processors A and B is not working.
      ANS: Simplification of A's report.

    Let $a =$"A is working," $b =$"B is working," and $c =$"C is working."

    (a) If you haven't already done so, write each status report in terms of a, b, and c, using the symbols of formal logic.
    ANS: A's report is $\neg b \wedge c$; B's report is $a \leftrightarrow b$; C's report is $\neg a \vee \neg b$.

    (b) How would you justify the conclusion that B is not working? (In other words, given the statements in part (a), which derivation rule allows you to conclude $\neg b$?)
    ANS: Simplification of A's report.

    (c) How would you justify the conclusion that C is working?
    ANS: Simplification of A's report.

    (d) Write a proof sequence to conclude that A is not working. (In other words, given the statements in part (a), write a proof sequence to conclude $\neg a$.)

Statements	Reasons
1. $\neg b \wedge c$	given (A's report)
2. $a \leftrightarrow b$	given (B's report)
3. $(a \to b) \wedge (b \to a)$	
exercise from 1.1 4. $a \to b$	simplification, 3
5. $\neg b$	simplification, 1
7. $\neg a$	*modus tollens*, 5, 4

23. (1.2#16) Write a proof sequence for the following assertion. Justify each step.

$$\left.\begin{array}{r} p \\ p \to r \\ q \to \neg r \end{array}\right\} \Rightarrow \neg q$$

Statements	Reasons
1. p	given
2. $p \to r$	given
3. $q \to \neg r$	given
4. r	*modus ponens*, 1,2
5. $\neg\neg r$	Double negation, 4
6. $\neg q$	*modus tollens*,5,3

24. (1.2#18) Write a proof sequence for the following assertion. Justify one of the steps in your proof using the result of Example 1.8.

$$\left.\begin{array}{r}\neg(a \wedge \neg b)\\ \neg b\end{array}\right\} \Rightarrow \neg\neg a$$

Statements	Reasons
1. $\neg(a \wedge \neg b)$	given
2. $\neg b$	given
3. $\neg a \vee \neg\neg b$	De Morgan's laws, 1
4. $\neg a \vee b$	double negation, 3
5. $b \vee \neg a$	commutivity, 4
6. $\neg a$	Example 1.10,5,2

25. (1.2#20) Write a proof sequence to establish that $p \Leftrightarrow p \vee p$ is a tautology.
ANS: All of the steps in the following proof sequence are reversible.

Statements	Reasons
1. $p \vee p$	given
2. $\neg\neg(p \vee p)$	double negation, 1
3. $\neg(\neg p \wedge \neg p)$	De Morgan's laws, 2
4. $\neg\neg p$	Simplification, 3
5. p	Double negation, 4

26. (1.2#22) Write a proof sequence for the following assertion. Justify each step.

$$(p \vee q) \vee (p \vee r) \Rightarrow \neg r \rightarrow (p \vee q)$$

Statements	Reasons
1. $(p \vee q) \vee (p \vee r)$	given
2. $(q \vee p) \vee (p \vee r)$	commutativity,1
3. $q \vee (p \vee (p \vee r))$	associativity, 2
4. $q \vee ((p \vee p) \vee r)$	commutativity, 3
5. $q \vee (p \vee r)$	Ecercise 20, 4
6. $(q \vee p) \vee r$	associativity, 5
7. $(p \vee q) \vee r$	commutativity, 6
8. $r \vee (p \vee q$	commutativity, 7
9. $\neg(\neg r) \vee (p \vee q)$	double negation, 8
10. $\neg r \rightarrow (p \vee q)$	implication, 9

27. (1.2#26) This exercise will lead you through a proof of the distributive property of $\wedge$ over $\vee$. We will prove:

$$p \wedge (q \vee r) \Rightarrow (p \wedge q) \vee (p \wedge r)$$

(a) The above assertion is the same as the following:

$$p \wedge (q \vee r) \Rightarrow \neg(p \wedge q) \rightarrow (p \wedge r)$$

Why? ANS: Implication.

(b) Use the deduction method from Exercise 25(c) to rewrite the tautology from part (a).
Problem 25 has you proving that $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$ with a sequence of equivalent statements like so:

Statements	Reasons
1. $p \to (q \to r)$	given
2. $\neg p \vee (q \to r)$	implication, 1
3. $\neg p \vee (\neg q \vee r)$	implication, 2
4. $(\neg p \vee \neg q) \vee 4$	associative, 3
5. $\neg(p \wedge q) \vee r$	de Morgan, 4
6. $(p \wedge q) \to r$	implication 5

in which each step is an "iff" step, so we have the tautology, $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$.
Thus, to prove $A \Rightarrow B \to C$ it is sufficient to prove

$$\left.\begin{matrix} A \\ B \end{matrix}\right\} \Rightarrow C$$

instead. Use this fact to rewrite the tautology

$$p \wedge (q \to r) \Rightarrow q \to (p \wedge r)$$

as a tautology of the form

$$\left.\begin{matrix} A \\ B \end{matrix}\right\} \Rightarrow C$$

where C does not contain the $\to$ connective. (The process of rewriting a tautology this way is called the *deductive method.*)
ANS: This is easy enough, as they have laid out all the steps and we simply write

$$\left.\begin{matrix} p \wedge (q \to r) \\ \neg(p \wedge q) \end{matrix}\right\} \Rightarrow p \wedge r$$

(c) Prove your rewritten tautology.
Go back and look at the proof given in ex. 6. This is exactly the same!

28. (1.2#28) Is $a \to \neg a$ a contradiction? Why or why not?

ANS: It is not a contradiction. The truth table is not always false:

a	$\neg a$	$a \to \neg a$
T	F	F
F	T	T