Math 15-Spring '17-Final Exam Solutions

1. Consider the following definition of the "$\lhd$" symbol.

   Definition. Let x and y be integers. Write $x \lhd y$ if $5x + 7y = 11k$ for some integer k.

   (a) Show that $1 \lhd 4, 2 \lhd 8$, and $4 \lhd 5$.
   ANS: $5 \cdot 1 + 7 \cdot 4 = 11 \cdot 3, 5 \cdot 2 + 7 \cdot 8 = 11 \cdot 6$, and $5 \cdot 4 + 7 \cdot 5 = 11 \cdot 5$

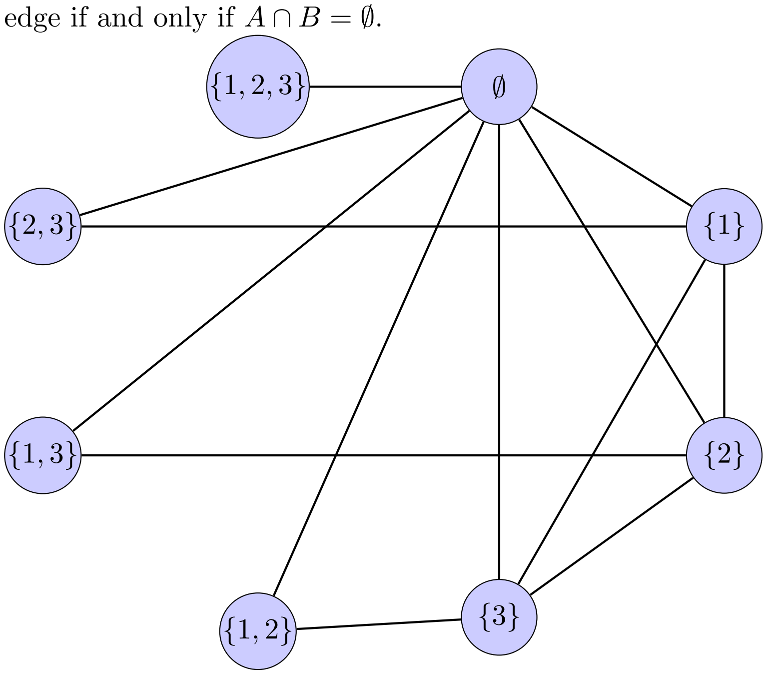
   (b) Find a counterexample to the following statement: If $a \lhd b$ and $c \lhd d$, then $a \cdot c \lhd b \cdot d$.
   ANS: $1 \lhd 4$ and $4 \lhd 5$, but $4 \not\lhd 20$, since $5 \cdot 4 + 7 \cdot 20 = 160$ is not a multiple of 11.

2. To prove a statement of the form $(\forall x)(P(x) \to Q(x))$, using contraposition, begin your proof with a sentence of the form

   Let x be [an element of the domain], and suppose $\neg Q(x)$.

   Then write a sequence of justified conclusions culminating in $\neg P(x)$.
   Use this method to prove the following:
   Let n be an integer. If $n^2 + n + 1$ is even, then n is odd.
   ANS: Let $n \in \mathbb{Z}$ and suppose $n \in 2\mathbb{Z}$, then $\exists k \in \mathbb{Z} \ni n = 2k$. This means that $n^2 + n = (2k)^2 + 2k = 2(2k^2 + k)$ is even, so $n^2 + n + 1$ is odd.
   The odd thing about this proof, of course, is that $n^2 + n + 1$ is always odd!

3. Draw an undirected graph G with the following properties. The vertices of G correspond to the elements of the power set $P(\{0, 1, 2\})$. Two vertices (corresponding to $A, B \in P(\{0, 1, 2\})$) are connected by an edge if and only if $A \cap B = \emptyset$.



4. Construct addition and multiplication tables for $\mathbb{Z}/7$. (only use equivalence class representatives in $\{0, 1, 2, 3, 4, 5, 6\}$)

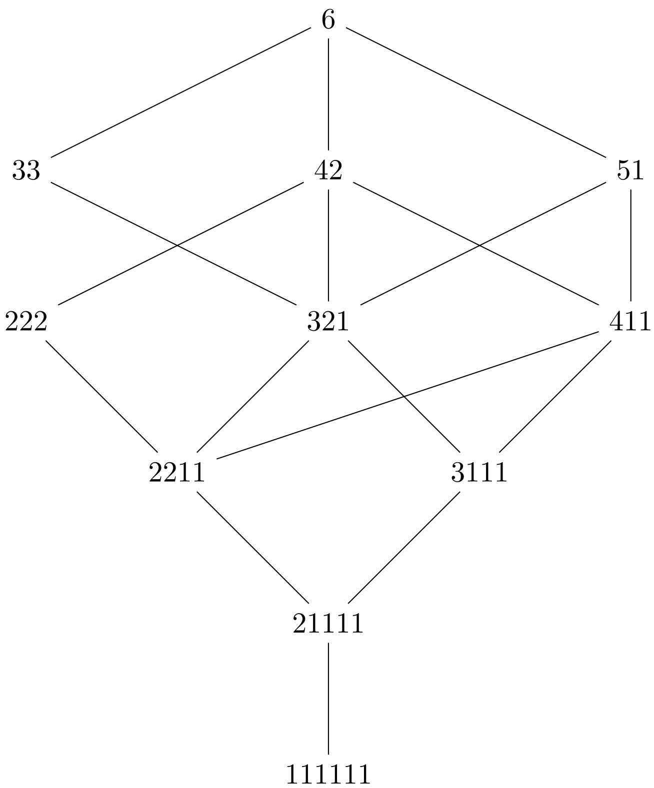
ANS:

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

·	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

5. A partition of a positive integer n is a list of positive integers $a_1, a_2, \ldots, a_k$ such that $a_1+a_2+\cdots+a_k = n$. For example, the partitions of 5 are $\{5\}, \{1,4\}, \{2,3\}, \{1,1,3\}, \{1,2,2\}, \{1,1,1,2\}, \{1,1,1,1,1\}$. The order of the list doesn't matter. There is a natural partial ordering on the set of partitions of n: if P_1 and P_2 are partitions, define $P_1 \preceq P_2$ if P_1 can be obtained by combining parts of P_2. For example, $\{1,2,2\} \preceq \{1,1,1,1,1\}$ because $\{1,2,2\} = \{1, 1+1, 1+1\}$. On the other hand, $\{2,3\}$ and $\{1,4\}$ are incomparable elements in this poset.

(a) Write the partitions of 6 in a Hasse diagram. (There are 11 partitions of 6.)



ANS:

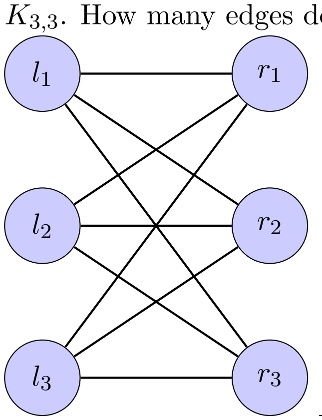
(b) If a poset $(X, \preceq)$ has no incomparable elements, it is called a total ordering. For example, the real numbers $\mathbb{R}$ are totally ordered by the $\leq$ relation. Is the above ordering on the partitions of 6 a total ordering? Why or why not?

ANS: No. There are incomparable elements, (e.g., 3,3 and 2,2,2).

6. The complete bipartite graph $K_{m,n}$ is the simple undirected graph with $m+n$ vertices split into two sets V_1 and V_2 ($|V_1| = m$, $|V_2| = n$) such that vertices x, y share an edge if and only if $x \in V_1$ and $y \in V_2$.

(a) Draw $K_{3,3}$. How many edges does it have?

ANS:



$K_{3,3}$ has 9 edges.

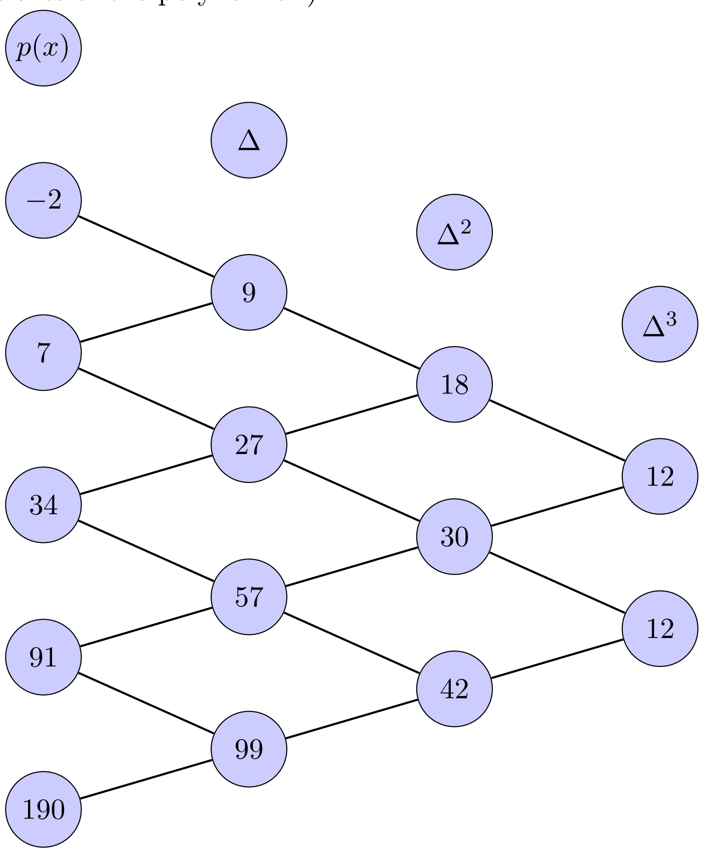
(b) How many edges does $K_{n,n}$ have? How do you know?

ANS: Since forming an edge means choosing 1 of the vertices on the left and 1 of the vertices on the right, and there are n choices for each, the number of edges is $n \cdot n = n^2$.

(c) Let $E(n)$ be the number of edges in $K_{n,n}$. Clearly, $E(1) = 1$. Write $E(n)$ in terms of $E(n-1)$ for $n > 1$.
ANS: $E(n) = E(n-1) + 2n - 1$ For instance, $E(3) = E(2) + 5 = 4 + 5$

7. Analyze the sequence $-2, 7, 34, 91, 190$ using sequences of differences.

(a) From what degree polynomial does this sequence appear to be drawn? (Don't bother finding the coefficients of the polynomial.)



ANS:
Evidently, the third-order difference are the same, so the polynomial is cubic.

(b) What would the next number in the sequence be?
Given that the third order difference of the next term will be 12, the second order difference would be 54, the first order difference would be 153 and $p(x_{\texttt{next}}) = 243$.

8. Define a set X of numbers as follows.
B. $5 \in X$.
R_1. If $x \in X$, so is $3x$. R_2. If $x \in X$, so is $x + 6$.

(a) List, in order, all the elements of X that are less than 50.
ANS: $\{x | x \in X \texttt{ and } x < 50\} = \{5, 11, 15, 17, 21, 23, 27, 29, 33, 35, 39, 41, 45, 47\}$

(b) Use induction to prove that every element of X is odd.
Base case: 5 is odd.
Assume that $x \in X$ is odd, then $x = 2k + 1$ for some $k \in \mathbb{Z}$ both $x + 6 = 2(k + 3) + 1$ and $3x = 2(3k + 1) + 1$ are odd, so all elements of X are odd.

9. Recall the definition for a sorted list:
Definition An `SList` is
B. x where $x \in \mathbb{R}$, the real numbers.
R. (X, Y) where X and Y are `SLists` having the same number of elements, and the last number in X

is less than the first number in Y.
Let L be an SList. Define a recursive function Dubup as follows.
B. Suppose $L = x$. Then $\texttt{Dubup}(L) = 2x+1$. **R**. Suppose $L = (X, Y)$. Then $\texttt{Dubup}(L) = \texttt{Dubup}(X) + \texttt{Dubup}(Y)$.

(a) Evaluate $\texttt{Dubup}(((1,2),(3,4)))$, showing all steps.
ANS: $\texttt{Dubup}(((1,2),(3,4))) = \texttt{Dubup}((1,2)) + \texttt{Dubup}((3,4)) = \texttt{Dubup}(1) + \texttt{Dubup}(2) + \texttt{Dubup}(3) + \texttt{Dubup}(4) = (2+1) + (4+1) + (6+1) + (8+1) = 24$

(b) Give a recurrence relation for $S(p)$, the number of $+$ operations performed by Dubup on an SList of depth p, for $p \geq 0$.
ANS:

$$S(p) = \begin{cases} 1 & : p = 0 \\ 2S(p-1) + 1 & : p > 1 \end{cases}$$

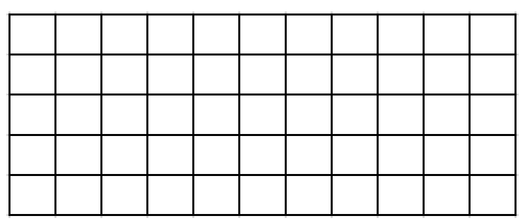
(c) Give a recurrence relation for $M(p)$, the number of $\cdot$ operations performed by Dubup on an SList of depth p, for $p \geq 0$.
ANS:

$$M(p) = \begin{cases} 1 & : p = 0 \\ 2M(p-1) & : p > 1 \end{cases}$$

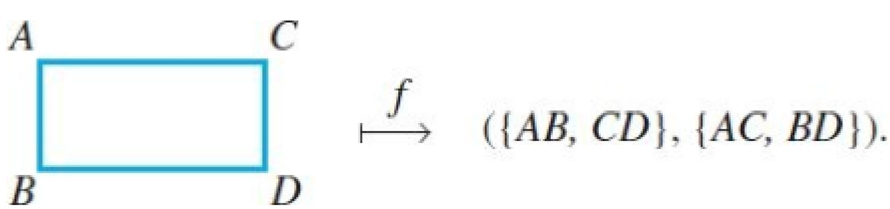
10. Suppose that license plates in a certain municipality come in two forms: one digit $(0 \ldots 9)$ followed by three letters $(A \ldots Z)$ followed by three digits. How many different license plates are possible? (You can express this as a product and not multiply it out.)
ANS: $10^4 \cdot 26^3 = 175,760,000$

11. How many different strings can be formed by rearranging the letters of "ABRACADABRA"? (Again, express this as a product without multiplying.)
ANS: $\dfrac{11!}{5! \cdot 2 \cdot 2} = 83160$

12. The following figure consists of 6 horizontal lines and 12 vertical lines. The goal of this problem is to count the number of rectangles (squares are a kind of rectangle, but line segments are not).



Let V be the set of all sets of two vertical lines, and let H be the set of all sets of two horizontal lines. Let R be the set of all rectangles in the figure. Define a function $f : R \to V \times H$ by



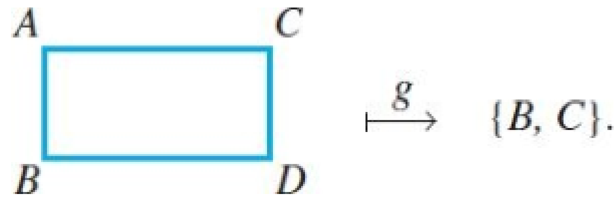
(a) Explain why f is well defined.
ANS: Two vertical lines and two horizontal lines completely determine a rectangle in this scheme. Plus, all possible pairs of vertical and horizontal lines are in the domain.

(b) Explain why f is one-to-one.
ANS: The function f is 1-1 (injective) because two vertical and two horizontal lines determine a unique rectangle.

(c) Explain why f is onto.
ANS: The function f is onto (surjective) because there is a rectangle for every two vertical and two horizontal lines: Every pair of vertical and horizontal lines is determined by some rectangle.

(d) Compute $|R|$, the number of rectangles in the figure.
ANS: $\binom{12}{2} \cdot \binom{6}{2} = 66 \cdot 15 = 990$.

(e) The previous figure contains $6 \cdot 12 = 72$ intersection points. Let P be the set of all sets $\{X, Y\}$ of two intersection points. Suppose we tried counting R by defining a function $g : R \to P$, which maps a rectangle to the set containing its lower left and upper right vertices:



Explain why g is not a one-to-one correspondence.
ANS: The function g is not onto (surjective): for example, if X is on the same vertical line as Y, there is no rectangle mapping to $\{X, Y\}$.

13. If you roll three four-sided dice (numbered 1,2,3, and 4), what is the probability of rolling a 5?
ANS: There are 4^3 equally-likely outcomes. 113, 131, 311, 221, 212, 122 are the only ways to get a 5, so the probability is $\frac{6}{64} = \frac{3}{32} = 0.09375$. To verify this experimentally (empirically), run the Python script:

```
import random
n=3
m=10000000
cnt = 0
for i in range(m):
    L = [random.randrange(1,5) for i in range(n)]
    #print(L)
    if sum(L)==5:
        cnt += 1
print(cnt/m)
```

which should, eventually (it's ten million iterations!) produce a result like 0.093678

14. Consider the following algorithm::
$x \leftarrow 1$
for $i \in \{1, 2, 3, 4\}$ do
    for $j \in \{1, 2, 3, 4, 5\}$ do
        $x \leftarrow x + x$
    for $k \in \{1, 2, 3, 4, 5, 6\}$ do
        $x \leftarrow x + 2$
        $x \leftarrow x + 3$

(a) Count the number of + operations done by this algorithm.
ANS: The first nested for-loop executes $4 \cdot 5 = 20$ additions and the next executes $4 \cdot 6 \cdot 2 = 48$ additions, for a total of 68 additions.

(b) What is the value of x after the algorithm finishes?
It's easier to track the result of this algorithm by simplifying it first:
$x \leftarrow 1$

```
for i ∈ {1,2,3,4} do
     for j ∈ {1,2,3,4,5}  do
         x ← 2x
         x ← x + 30
```

The inner for-loop then is first doubling and then adding 30, five times:
$f(x) = 2(2(2(2(2x + 30) + 30) + 30) + 30) + 30$. The first time through we get $f(1) : 32 \to 94 \to 218 \to 466 \to 962$, but then we loop it three more times:
$962 \to 1954 \to 3938 \to 7906 \to 15842 \to 31714$ and again:
$31714 \to 63458 \to 126946 \to 253922 \to 507874 \to 1,015,778$, and finally:
$1,015,778 \to 2,031,586 \to 4,063,202 \to 8,126,434 \to 16,252,898 \to 32,505,826$.
There's a nice round number...perhaps a little Python script, to be sure:

```python
x=1
for i in range(4):
    print(i,x)
    for j in range(5):
        x = x+x
        for k in range(6):
            x += 5
x
```

produces

```
0 1
1 962
2 31714
3 1015778
Out[14]:
32505826
```

15. Let $f : \mathbb{N} \to \mathbb{R}$ be a function. Then $O(f)$ is the set of all functions g such that

$$g(n) \leq Kf(n)$$

for some constant $K > 0$, and for all $n \geq N$ for some $N > 0$. If $g \in O(f)$, we also say that "g is big-oh of f." Let $p, q \in \mathbb{N}$ with $0 < p < q$. Show that $n^p \in O(n^q)$
ANS: Choose $K = 1$ and $N = 1$, then $g(n) = n^p = \dfrac{n^q}{n^{q-p}} \leq n^q = f(n)$ since $q - p > 0$ and $n > 1$.

16. Write a recursive function in pseudocode that computes the value of the following recurrence relation:

$$T(n) = \begin{cases} 1 & : n = 1 \\ T(n-1) + n & : n > 1 \end{cases}$$

ANS:

```
function T(n ∈ mathbbZ)
    if n = 1
        return 1
    else return T(n − 1) + n
```

or, in Python,

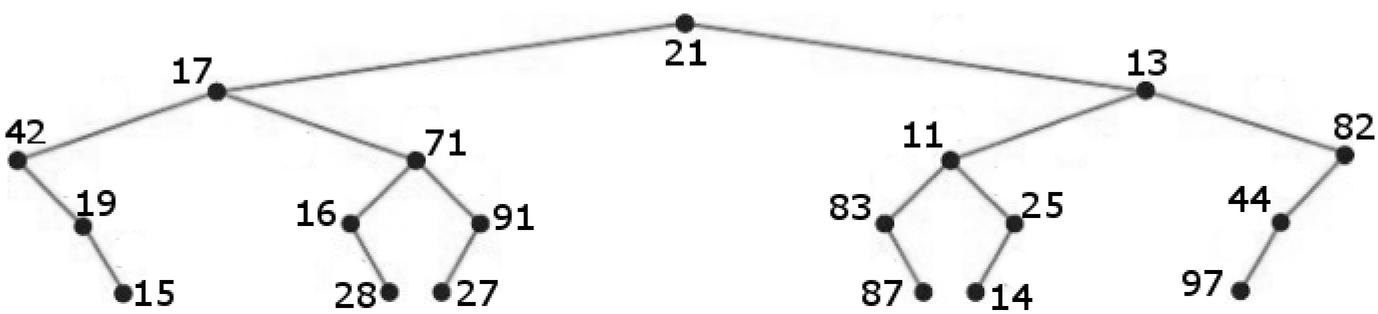
```python
def T(n):
    if n == 1:
        return 1
    else:
```

```python
        return T(n-1)+n
for n in range(1,10):
    print('T(',n,')=',T(n))
```

```
T( 1 )= 1
T( 2 )= 3
T( 3 )= 6
T( 4 )= 10
T( 5 )= 15
T( 6 )= 21
T( 7 )= 28
T( 8 )= 36
T( 9 )= 45
```

The triangular numbers.

17. Let T be a binary tree shown below.



(a) Suppose an algorithm uses a preorder traversal `PreOrder`$(T, 17)$. What sequence of nodes would the algorithm traverse?
ANS: I'll take advantage of some extra time and access to Python to do this. First, I enter the tree as a list like so:

```
T = [21,   #root
      [17, #left subtree
       [42,[],
        [19,[],[15,[],[]]]],
       [71,
        [16,[],[28,[],[]]],
        [91,[27,[],[]],[]]]],
      [13, #right subtree
       [11,
        [83,[],[87,[],[]]],
        [25,[14,[],[]],[]]],
       [82,[44,[97,[],[]],[]],[]]]]
```

Then I run the Python code for preorder traversal that we wrote in class, including a `print()` statement for each visitation:

```python
def PreSearch(t,T):
    if len(T)==0:
        return False
    elif t==T[0]:
```

```
            return True
        else:
            print(T[0],end=',')
            return PreSearch(t,T[1]) or PreSearch(t,T[2])
print(PreSearch(88,T))
```

Here's the preorder traversal:

`21,17,42,19,15,71,16,28,91,27,13,11,83,87,25,14,82,44,97`

(b) Suppose the algorithm uses an inorder traversal `InOrder`$(71, T)$. What sequence of nodes would the algorithm traverse?
ANS: Again, I'll use Python:

```
def PostSearch(t,T):
    if len(T)==0:
        return False
    elif PostSearch(t,T[1]) or PostSearch(t,T[2]):
        return True
    else:
        print(T[0],end=',')
        return t==T[0]
print(PostSearch(71,T))
```

Returns

`15,19,42,28,16,27,91,71,True`

18. Is there an assignment of true/false values for the variables p, q, r, s, and t such that the formula

$$[(p \vee q) \wedge (\neg p \vee \neg r)] \rightarrow [(s \vee t) \wedge (\neg s \vee \neg t)]$$

takes the value true? If so, find such an assignment. If not, explain why not.
ANS: Any assignment of values where p and q are both false will make the value of the formula true. The idea is that a false premis allows any conclusion at all. For example, If James Earl Ray shot MLK, then the cow jumps over the moon.