Math 15-Spring '17-Final Exam Show detailed work for credit.

1. Consider the following definition of the "$\lhd$" symbol.

   Definition. Let x and y be integers. Write $x \lhd y$ if $5x + 7y = 11k$ for some integer k.

   (a) Show that $1 \lhd 4, 2 \lhd 8$, and $4 \lhd 5$.

   (b) Find a counterexample to the following statement: If $a \lhd b$ and $c \lhd d$, then $a \cdot c \lhd b \cdot d$.

2. To prove a statement of the form $(\forall x)(P(x) \to Q(x))$, using contraposition, begin your proof with a sentence of the form

   Let x be [an element of the domain], and suppose $\neg Q(x)$.

   Then write a sequence of justified conclusions culminating in $\neg P(x)$.
   Use this method to prove the following:
   Let n be an integer. If $n^2 + n + 1$ is even, then n is odd.

3. Draw an undirected graph G with the following properties. The vertices of G correspond to the elements of the power set $P(\{0, 1, 2\})$. Two vertices (corresponding to $A, B \in P(\{0, 1, 2\})$) are connected by an edge if and only if $A \cap B = \emptyset$.

4. Construct addition and multiplication tables for $\mathbb{Z}/7$. (only use equivalence class representatives in $\{0, 1, 2, 3, 4, 5, 6\}$)

5. A partition of a positive integer n is a list of positive integers $a_1, a_2, \ldots, a_k$ such that $a_1+a_2+\cdots+a_k = n$. For example, the partitions of 5 are $\{5\}, \{1, 4\}, \{2, 3\}, \{1, 1, 3\}, \{1, 2, 2\}, \{1, 1, 1, 2\}, \{1, 1, 1, 1, 1\}$. The order of the list doesn't matter. There is a natural partial ordering on the set of partitions of n: if P_1 and P_2 are partitions, define $P_1 \preceq P_2$ if P_1 can be obtained by combining parts of P_2. For example, $\{1, 2, 2\} \preceq \{1, 1, 1, 1, 1\}$ because $\{1, 2, 2\} = \{1, 1 + 1, 1 + 1\}$. On the other hand, $\{2, 3\}$ and $\{1, 4\}$ are incomparable elements in this poset.

   (a) Write the partitions of 6 in a Hasse diagram. (There are 11 partitions of 6.)

   (b) If a poset $(X, \preceq)$ has no incomparable elements, it is called a total ordering. For example, the real numbers $\mathbb{R}$ are totally ordered by the $\leq$ relation. Is the above ordering on the partitions of 6 a total ordering? Why or why not?

6. The complete bipartite graph $K_{m,n}$ is the simple undirected graph with $m + n$ vertices split into two sets V_1 and V_2 ($|V_1| = m$, $|V_2| = n$) such that vertices x, y share an edge if and only if $x \in V_1$ and $y \in V_2$.

   (a) Draw $K_{3,3}$. How many edges does it have?

   (b) How many edges does $K_{n,n}$ have? How do you know?

   (c) Let $E(n)$ be the number of edges in $K_{n,n}$. Clearly, $E(1) = 1$. Wrote $E(n)$ in terms of $E(n - 1)$ for $n > 1$.

7. Analyze the sequence $-2, 7, 34, 91, 190$ using sequences of differences.

   (a) From what degree polynomial does this sequence appear to be drawn? (Don't bother finding the coefficients of the polynomial.)

   (b) What would the next number in the sequence be?

8. Define a set X of numbers as follows.
   B. $5 \in X$.
   R_1. If $x \in X$, so is $3x$. R_2. If $x \in X$, so is $x + 6$.

   (a) List, in order, all the elements of X that are less than 50.

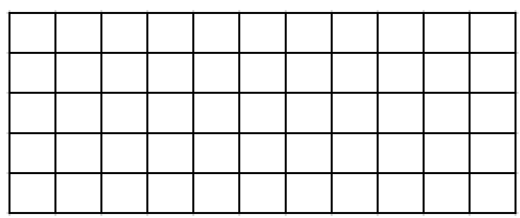
   (b) Use induction to prove that every element of X is odd.

9. Recall the definition for a sorted list:
Definition An `SList` is
B. x where $x \in \mathbb{R}$, the real numbers.
R. (X, Y) where X and Y are `SLists` having the same number of elements, and the last number in X is less than the first number in Y.
Let L be an `SList`. Define a recursive function `Dubup` as follows.
B. Suppose $L = x$. Then $\texttt{Dubup}(L) = 2x{+}1$. **R**. Suppose $L = (X, Y)$. Then $\texttt{Dubup}(L) = \texttt{Dubup}(X){+}\texttt{Dubup}(Y)$.
   (a) Evaluate $\texttt{Dubup}(((1, 2), (3, 4)))$, showing all steps.
   (b) Give a recurrence relation for $S(p)$, the number of $+$ operations performed by `Dubup` on an `SList` of depth p, for $p \geq 0$.
   (c) Give a recurrence relation for $M(p)$, the number of $\cdot$ operations performed by `Dubup` on an `SList` of depth p, for $p \geq 0$.

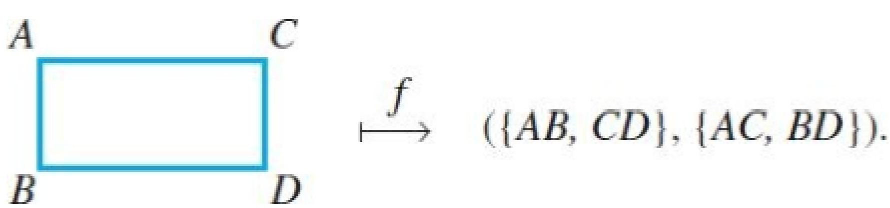
10. Suppose that license plates in a certain municipality come in two forms: one digit $(0 \ldots 9)$ followed by three letters $(A \ldots Z)$ followed by three digits. How many different license plates are possible? (You can express this as a product and not multiply it out.)

11. How many different strings can be formed by rearranging the letters of "ABRACADABRA"? (Again, express this as a product without multiplying.)

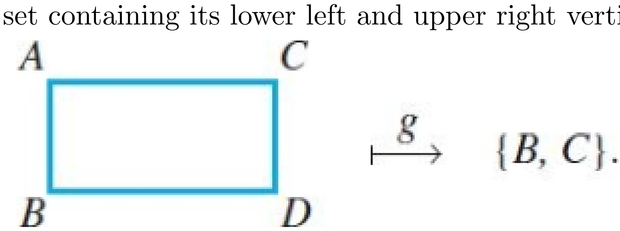
12. The following figure consists of 6 horizontal lines and 12 vertical lines. The goal of this problem is to count the number of rectangles (squares are a kind of rectangle, but line segments are not).



Let V be the set of all sets of two vertical lines, and let H be the set of all sets of two horizontal lines. Let R be the set of all rectangles in the figure. Define a function $f : R \to V \times H$ by



   (a) Explain why f is well defined.
   (b) Explain why f is one-to-one.
   (c) Explain why f is onto.
   (d) Compute $|R|$, the number of rectangles in the figure.
   (e) The previous figure contains $6 \cdot 12 = 72$ intersection points. Let P be the set of all sets $\{X, Y\}$ of two intersection points. Suppose we tried counting R by defining a function $g : R \to P$, which maps a rectangle to the set containing its lower left and upper right vertices:



Explain why g is not a one-to-one correspondence.

13. If you roll three four-sided dice (numbered 1,2,3, and 4), what is the probability of rolling a 5?

14. Consider the following algorithm.

```
x ← 1
for i ∈ {1, 2, 3, 4} do
     for j ∈ {1, 2, 3, 4, 5}  do
          x ← x + x
     for k ∈ {1, 2, 3, 4, 5, 6}  do
          x ← x + 2
          x ← x + 3
```

   (a) Count the number of + operations done by this algorithm.

   (b) What is the value of x after the algorithm finishes?

15. Let $f : \mathbb{N} \to \mathbb{R}$ be a function. Then $O(f)$ is the set of all functions g such that

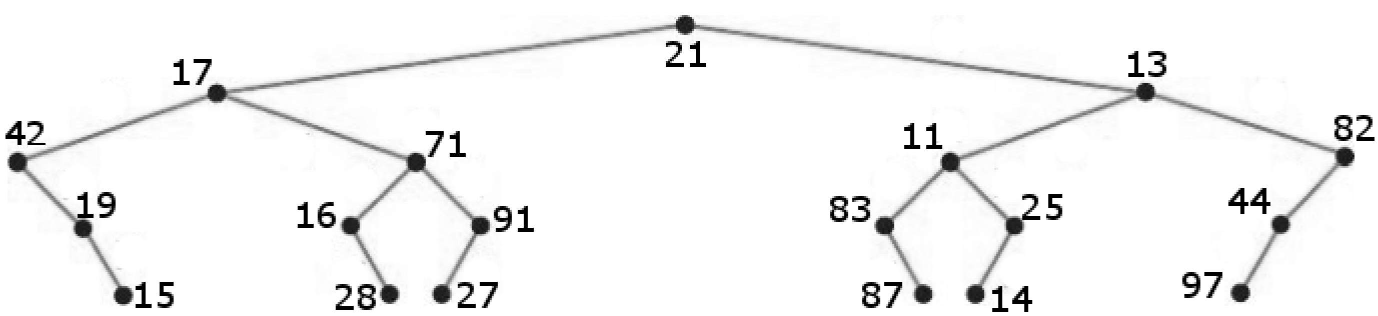
$$g(n) \leq Kf(n)$$

for some constant $K > 0$, and for all $n \geq N$ for some $N > 0$. If $g \in O(f)$, we also say that "g is big-oh of f." Let $p, q \in \mathbb{N}$ with $0 < p < q$. Show that $n^p \in O(n^q)$

16. Write a recursive function in pseudocode that computes the value of the following recurrence relation:

$$T(n) = \begin{cases} 1 & : n = 1 \\ T(n-1) + n & : n > 1 \end{cases}$$

17. Let T be a binary tree shown below.



   (a) Suppose an algorithm uses a preorder traversal `PreOrder`$(T, 17)$. What sequence of nodes would the algorithm traverse?

   (b) Suppose the algorithm uses an inorder traversal `InOrder`$(T, 71)$. What sequence of nodes would the algorithm traverse?

18. Is there an assignment of true/false values for the variables p, q, r, s, and t such that the formula

$$[(p \vee q) \wedge (\neg p \vee \neg r)] \to [(s \vee t) \wedge (\neg s \vee \neg t)]$$

takes the value true? If so, find such an assignment. If not, explain why not.