Math 15 - Spring '17 – Chapters 1 and 2 Test Solutions

1. Consider the declaratives statements,

   P: The moon shares nothing. Q: It is the sun that shares our works. R: The earth is alive with creeping men.

   Using logical connectives, write a proposition which symbolizes the following:

   (a) If the moon shares nothing and it is the sun that shares our works, then the earth is alive with creeping men.
   ANS: $P \wedge Q \to R$

   (b) The earth is alive with creeping men only if the moon shares nothing.
   ANS: ANS: This is equivalent to saying, "If the moon shares nothing, then the earth is alive with creeping men. That is, P is necessary for R or, in symbols, $P \Rightarrow R$

   (c) If the moon shares everything or the sun does not share our works then the the earth is not alive with creeping men.
   ANS: $\neg P \vee \neg Q \to \neg R \Leftrightarrow \neg(P \wedge Q) \to \neg R \Leftrightarrow P \wedge Q \vee \neg R$

   (d) If the moon shares nothing and the sun does not share our works, then the earth is alive with creeping men.
   ANS: $P \wedge \neg Q \to R$

2. Determine which of the following statements is true. _ Exactly one of these statements is false.
   _ Exactly two of these statements are false.
   _ Exactly three of these statements are false.
   _ Exactly four of these statements are false.
   _ Exactly five of these statements are false.
   _ Exactly six of these statements are false.
   _ Exactly seven of these statements are false.
   _ Exactly eight of these statements are false.
   _ Exactly nine of these statements are false.
   _ Exactly ten of these statements are false.
   ANS: Only "Exactly nine of these statements are false" can be true.

3. Construct a truth table for the proposition $P = [p \to (q \vee r)] \wedge \neg(p \leftrightarrow \neg r)]$.

p	q	r	$q \vee r$	$p \to (q \vee r)$	$\neg r$	$p \leftrightarrow \neg r$	$\neg(p \leftrightarrow \neg r)$	P
0	0	0	0	1	1	0	1	1
0	0	1	1	1	0	1	0	0
0	1	0	1	1	1	0	1	1
0	1	1	1	1	0	1	0	0
1	0	0	0	0	1	1	0	0
1	0	1	1	1	0	0	1	1
1	1	0	1	1	1	1	0	0
1	1	1	1	1	0	0	1	1

4. Suppose we know A is true, $A \to (B \to C)$ is true and that $B \to D$ is true. Can we conclude that $\neg C \to D$? Prove your claim.

   ANS: We are assuming that A is true, so $A \to (B \to C)$ is equivalent to, and can be simplified as $B \to C$. Thus we can simplify our assumptions to $B \to C$ and $B \to D$. Using the implication rule, we can rewrite these as $\neg B \vee C$ and $\neg B \vee D$. Now, the implication rule also says that $\neg C \to \neg D$ is equivalent to $C \vee D$. See the truth table:

$\neg B$	C	D	$\neg B \vee C$	$\neg B \vee D$	$C \vee D$
0	1	1	1	1	1
1	0	0	1	1	0
1	1	1	1	1	1
1	0	1	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1

This shows that the conclusion does not follow, since in the case where B, C, and D are all false, $B \to C$ and $B \to D$ are both true, but $\neg C \to D$ is false. We could also do a deductive proof:

	statement	reason
1	A	Given
2	$A \to (B \to C)$	Given
3	$(B \to D$	Given
4	$B \to C$	Modus ponens (1 and 2)
5	$B \to (C \wedge D)$	addition (3 and 4)
6	$\neg B \vee (C \wedge D)$	implication (5)
7	$\neg B \vee C) \wedge (\neg B \vee D)$	Distributive property
8	$\neg C \to B) \wedge (\neg D \to C)$	Contrapositive (3 and 4)
9	and so	on
10	and	on

5. Given that $\neg A \to B$ and $B \to A$ and $A \to \neg B$ can we conclude $A \wedge \neg B$? Use the method of indirect proof to prove or disprove.

	statement	reason
1	$\neg A \to B$	Given
2	$B \to A$	Given
3	$A \to \neg B$	Given
4	$\neg A \to B$	Contrapositive (2)
5	$\neg(\neg A)$	absurdity (1 and 4)
6	A	double negative (5)
7	$B \to \neg A$	Contrapositive (3)
8	$\neg B$	Absuridty (2 and 7)
9	$A \wedge \neg B$	Addition (6 and 8)

6. Prove that implication is not associative.
ANS: That is, we want to show that $(A \to B) \to C$ is not equivalent to $A \to (B \to C)$. The implication rule tells us that $(A \to B) \to C$ is equivalent to $\neg(\neg A \vee B) \vee C$ which is equivalent to $(A \wedge (\neg B) \vee C$ by de Morgan's laws and distributing the "or", we have the equivalent statement $(A \vee C) \wedge (\neg B \vee C)$.
On the other hand, $A \to (B \to C)$ is equivalent to $\neg A \vee (\neg B \vee C) = \neg A \vee (\neg B \vee C$. To be sure, the last two columns of this truth table show these are not equivalent:

A	B	C	$A \vee C$	$\neg B \vee C$	$(A \vee C) \wedge (\neg B \vee C)$	$\neg A \vee \neg B \vee C$
0	0	0	0	1	0	1
0	0	1	1	1	1	1
0	1	0	0	0	0	1
0	1	1	1	1	1	1
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

7. Prove or disprove: $(A \to B) \vee (A \to C) \to B \vee C$ is equivalent to $(A \vee B \vee C)$.
ANS: The knucklehead approach is to do the truth table and see that their logic is the same.

A	B	C	$A \to B$	$A \to C$	$(A \to B) \vee (A \to C)$	$B \vee C$	$(A \to B) \vee (A \to C) \to B \vee C$
0	0	0	1	1	1	0	0
0	0	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	1
1	0	1	0	1	1	1	1
1	1	0	1	0	1	1	1
1	1	1	1	1	1	1	1

...and here is a two-column proof:

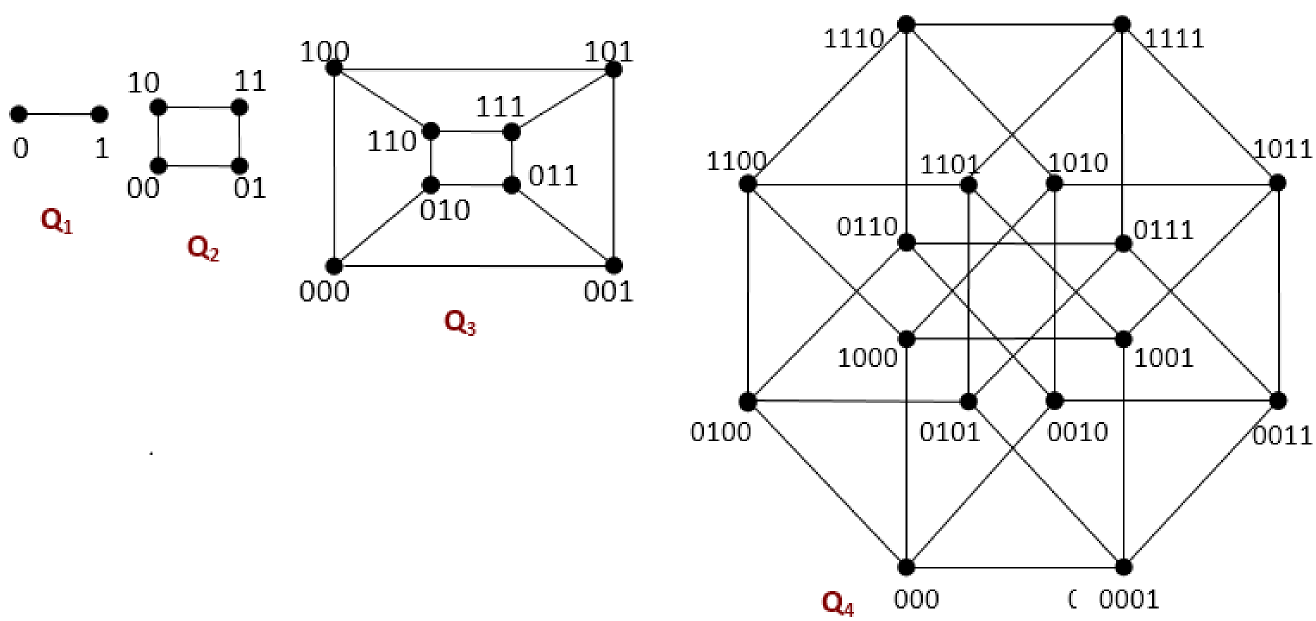
	statement	reason
1	$(A \to B) \vee (A \to C) \to B \vee C$	Given
2	$\neg((\neg A \vee B) \vee (\neg A \vee C)) \vee (B \vee C)$	Implication
3	$\neg(\neg(A \vee B) \wedge \neg(\neg A \vee C)) \vee (B \vee C)$	De Morgan (2)
4	$(A \wedge \neg B \wedge A \wedge \neg C) \vee (B \vee C)$	De Morgan (3)
5	$A \wedge \neg B \wedge \neg C) \vee (B \vee C)$	Simplification (4)
6	$A \vee (B \vee C) \wedge \neg B \vee (B \vee C) \wedge \neg C \vee (B \vee C)$	Distributive law (5)
7	$(A \vee B \vee C) \wedge (\neg B \vee B \vee C) \wedge (\neg C \vee B \vee C)$	Associative law (6)
8	$A \vee B \vee C$	Simplification (7)

8. First, translate the following argument into propositional logic. Then, prove that the argument is valid using the method of formal derivation. Explain your answer.

   Sally is destined to be either a fearless adventurer or a great psychiatrist. If Nabokov is not a great writer then Sally is not destined to be a great psychiatrist. Yet Sally is clearly not destined to be fearless adventurer. So Nabokov is definitely a great writer.

ANS: Let P: Sally is destined to be a fearless adventurer
Let Q: Sally is destined to be a great psychologist
Let N: Nabokov is a great writer. Given $P \wedge Q$, $\neg R \to \neg Q$, and $\neg P$ prove R

	statement	reason
1	$P \vee Q$	Given
2	$\neg N \to \neg Q$	Given
3	$\neg P$	Given
4	$Q \to R$	Contrapositive (2)
5	$\neg P \to Q$	Implication (1 and 4)
6	Q	Modus ponens (3 and 5)
7	N	Modus ponens (4 and 6)

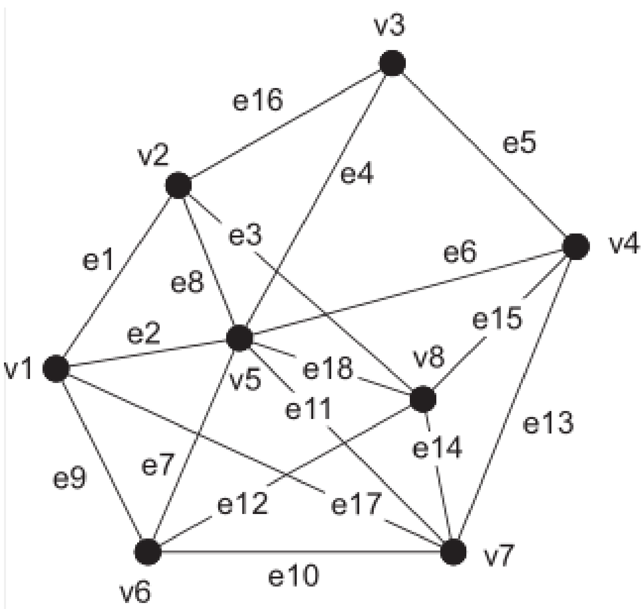


9. Of particular importance in coding theory are the hypercube graphs, which may be constructed by taking as vertices all binary words (sequences of 0s and 1s) of a given length and joining two of these vertices if the corresponding binary words differ in just one place. The graph obtained in this way from the binary words of length k is called the k-hypercube (or k-dimensional cube), and is denoted Q_k. Cube graphs for $k = 1, 2, 3$ and 4 are shown above. Give a formula for the number of edges of a Q_k graph.
ANS: The number of vertices of Q_k can be computed using the fundamental principle of counting or the formula for handshakes: The number of vertices is the number of k-digit binary numbers or $n = 2^k$. Since, by definition, the valence of each vertex is the number of digits, k, we have

$$\frac{k2^k}{2} = k2^{k-1}$$

edges.

10. Listing the vertex degrees of a graph gives us a **degree sequence**. The vertex degrees are usually listed in descending order, in which case we refer to an ordered degree sequence. For example, if we consider the eight vertices of graph G below,



$V(G) = \{v_1, \ldots, v_8\}$
$E(G) = \{e_1, \ldots, e_{18}\}$

$e_1 = \langle v_1, v_2 \rangle$	$e_{10} = \langle v_6, v_7 \rangle$
$e_2 = \langle v_1, v_5 \rangle$	$e_{11} = \langle v_5, v_7 \rangle$
$e_3 = \langle v_2, v_8 \rangle$	$e_{12} = \langle v_6, v_8 \rangle$
$e_4 = \langle v_3, v_5 \rangle$	$e_{13} = \langle v_4, v_7 \rangle$
$e_5 = \langle v_3, v_4 \rangle$	$e_{14} = \langle v_7, v_8 \rangle$
$e_6 = \langle v_4, v_5 \rangle$	$e_{15} = \langle v_4, v_8 \rangle$
$e_7 = \langle v_5, v_6 \rangle$	$e_{16} = \langle v_2, v_3 \rangle$
$e_8 = \langle v_2, v_5 \rangle$	$e_{17} = \langle v_1, v_7 \rangle$
$e_9 = \langle v_1, v_6 \rangle$	$e_{18} = \langle v_5, v_8 \rangle$

we have the following vertex degrees

vertex	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
degree	4	4	3	4	7	4	5	5

which, when ordering these degrees in descending order, leads to the ordered degree sequence

$$[7, 5, 5, 4, 4, 4, 4, 3]$$

If every vertex has the same degree, the graph is called regular. In a k-regular graph each vertex has degree k. As a special case, 3-regular graphs are also called **cubic graphs**.
If every vertex has the same degree, the graph is called regular. In a k-regular graph each vertex has degree k. As a special case, 3-regular graphs are also called **cubic graphs**.

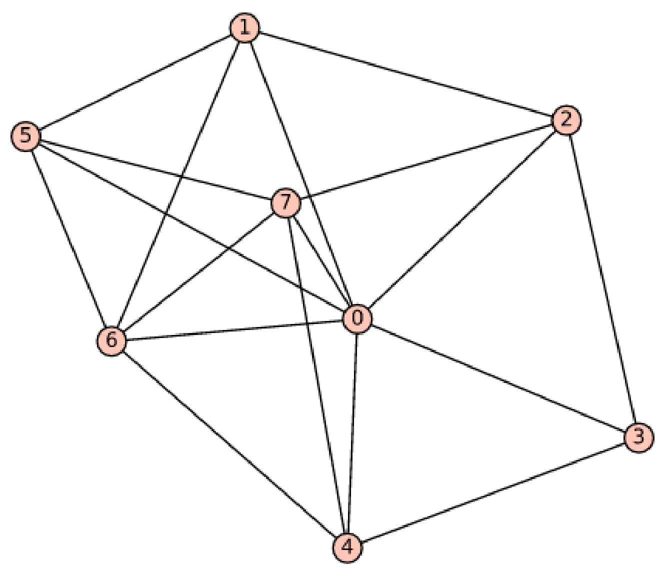
When considering degree sequences, it is common practice to focus only on simple graphs, that is, graphs without loops and multiple edges. An interesting question that comes to mind is when we are given a list of numbers, is there also a simple graph whose degree sequence corresponds to that list? There are some obvious cases where we already know that a given list cannot correspond to a degree sequence. For example, we have proven that the sum of vertex degrees is always even. Therefore, a minimal requirement is that the sum of the elements of that list should be even as well. Likewise, it is not difficult to see that, for example, the sequence $[4, 4, 3, 3]$ cannot correspond to a degree sequence. In this case, if this were a degree sequence, we would be dealing with a graph of four vertices. The first vertex is supposed to have four incident edges. In the case of simple graphs, each of these edges should be incident with a different vertex. However, there are only three vertices left to choose from, so $[4, 4, 3, 3]$ can never correspond to the degree sequence of a simple graph.

Does the degree list uniquely identify a graph? The list $[7, 5, 5, 4, 4, 4, 4, 3]$, for instance may not describe a unique graph, or any graph at all (if we pretend that we haven't already seen it for ourselves!) An algorithm for checking that a degree list describes an actual graph is to successively delete the highest degree vertex (and all its edges) from the graph and see if you maintain consistency with actual graphs.

In Sagemath, for instance, we can create the graph we started with (renaming the vertices) like so:

```
dg = {0: [1,2,3,4,5,6,7], 1: [2,5,6], 2: [3,7], 3: [4], 4: [6,7], \
5: [6,7], 6: [7] }
G1=Graph(dg)
G1.plot()
```

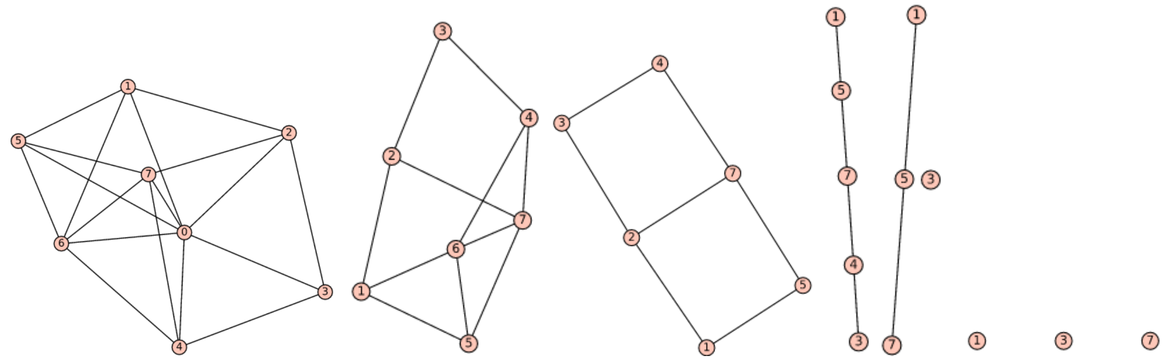
which produces this rendering:



Repeatedly applying the procedure

```
G1.delete_vertex(0);
G1.delete_vertex(6);
G1.delete_vertex(4);
G1.delete_vertex(2);
G1.delete_vertex(5);
```

produces a sequence of graphs:

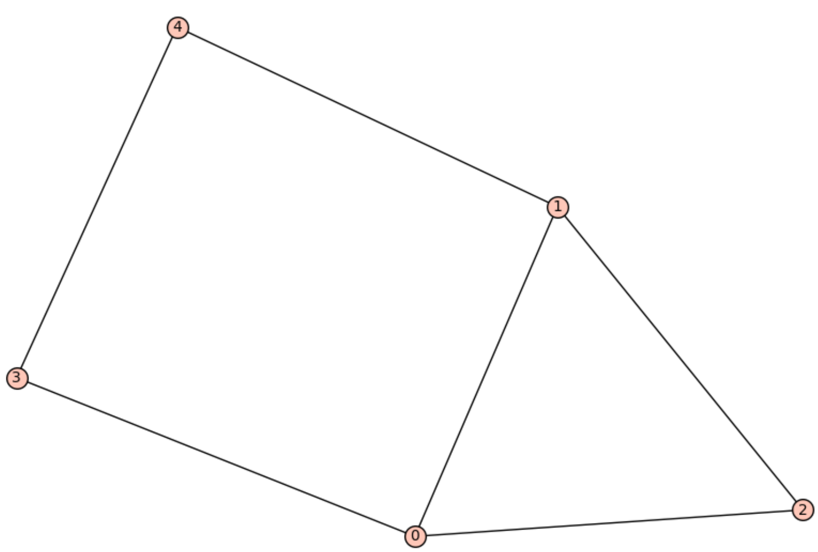


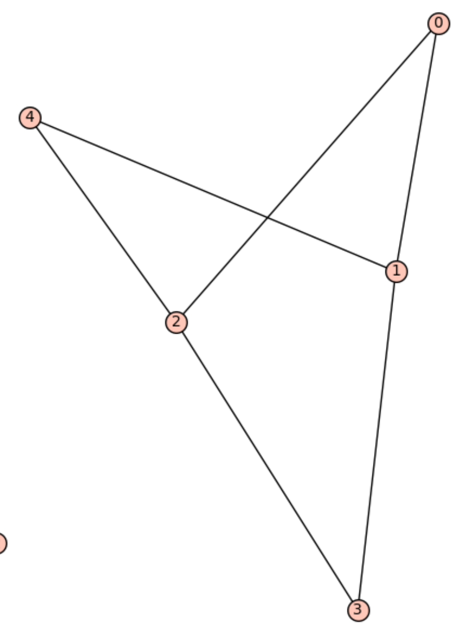
(a) Will the procedure for verifying a degree list work if we delete vertices in any order whatsoever? If so, why choose to delete the highest degree vertices first?
ANS: Yes, since the deleting of a vertex and its edges is always reversible (albeit, there may be more than one way to go backwards, there is always the place you came from as a place to go back to). The reason for deleting the highest degree vertices first is that it more quickly simplifies the graph.

(b) Construct two different graphs with the degree sequence $[3, 3, 2, 2, 2]$
ANS: Using the Sagemath commands

```
gdict1 = {0 : [1,2,3], 1 : [2,4], 3 : [4]}
G1 = Graph(gdict1)
G1.plot()
gdict2 = {0 : [1,2], 1 : [3,4], 2 : [3, 4]}
G2 = Graph(gdict2)
G2.plot()
```

we generate the two non-isomorphic graphs below:

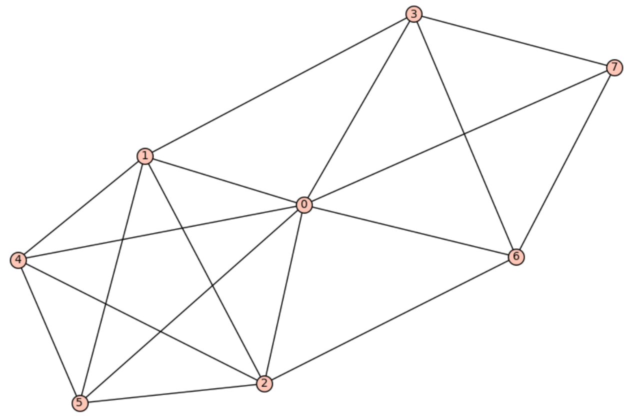




(c) Construct two different graphs with the degree sequence $[7, 5, 5, 4, 4, 4, 4, 3]$
Sagemath has a sequence of commands we can use to build a graph with this degree sequence:

```
ds = [7,5,5,4,4,4,4,3]
g = graphs.DegreeSequence(ds)
g.degree_sequence()
sage.graphs.graph_generators.graphs.DegreeSequence(ds)
```

This produces the graph shown at right:



Applying our algorithm to this graph gives us the sequence of graphs with degree sequences, $\texttt{ds}(G_1) = [7,5,5,4,4,4,4,3]$ It should be clear that if we do not change the ordering of vertex degrees, that the degree sequence of G_2 is equal to $[4,4,3,3,3,3,2]$. First, it contains one element less than the degree sequence of G_1 . Second, the first element of the degree sequence of G_2 corresponds to the second element of G_1's degree sequence: it's the degree of the same vertex, yet for G_2 it should be one less than in G_1 because this vertex is not yet joined to the added vertex v_1 . Likewise, the second element of G_2's degree sequence corresponds to the third one in the degree sequence of G_1, and so on.
If $[4,4,3,3,3,3,2]$ is graphic we can apply the same trick: G_2 should be constructable from a graph G_3 by adding a vertex v_2 and joining v_2 to four vertices from G_3. Following a completely analogous procedure as before, v_2 is joined to the vertices from G_3 such that these vertices will then have vertex degree 4, 3, 3, and 3, respectively. This can only mean that in G_3 they will have degree 3, 2, 2, and 2, respectively, leading to the following list: $[3,2,2,2,3,2]$.

Note that in this example, the fifth element is the same as the sixth element in the degree sequence of G_2. The first four elements represent vertices that will be joined to the new vertex v_2. The other elements represent vertices that remain untouched, and will thus have the same number of incident edges in G_2.

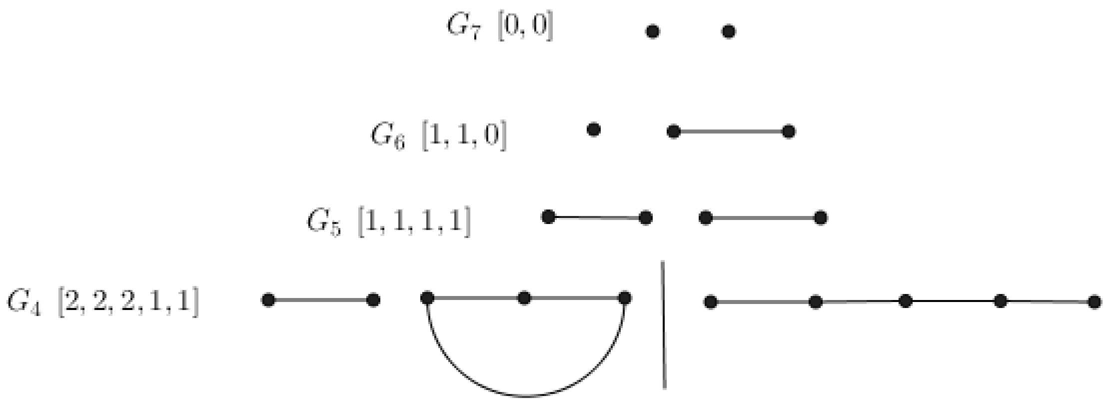
Continuing this line of reasoning, if $[3,3,2,2,2,2]$ is the (now ordered) degree sequence of G_3, then we should be able to construct G_3 from a graph G_4 to which we have added a vertex v_3. This vertex would be joined to the vertices having degree 2, 1, and 1 in G_4, respectively, yielding the list $[2,1,1,2,2]$. Again, note that this list contains one element less than the degree sequence of G_3, but that now its fourth and subsequent elements represent vertices that have the same vertex degree in G_4 and G_3.
We now have that if ordered list $[2,2,2,1,1]$ is graphic, then so should $[1,1,1,1]$, corresponding to a graph G_5.

Likewise, if $[1,1,1,1]$ is graphic, then so should the list of vertex degrees $[0,1,1]$ correspond to a graph G_6.

Finally, if the ordered list $[1,1,0]$ is graphic, then so should $[0,0]$, which is true: it is a graph G_7 with two vertices and no edges.

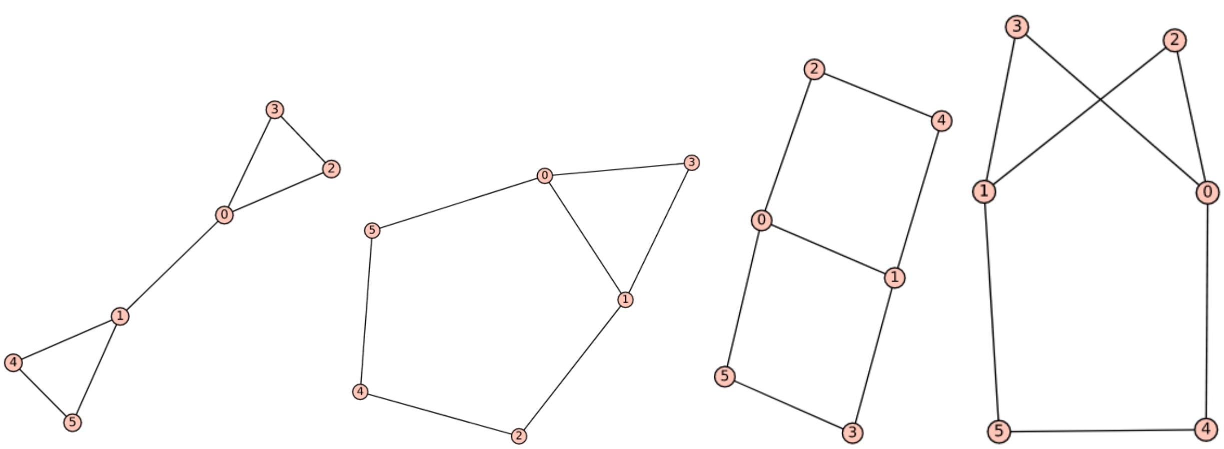
We can safely conclude that the sequence $[7,5,5,4,4,4,4,3]$ indeed corresponds to a simple graph. The construction of the graph G_1 is illustrated below, where each graph $G_1, G_2, \ldots, G_6$ is constructed by adding a vertex to the previous one, starting from graph G_7 . The question of whether G_1 is the same as the graph from Sagmeath's figure remains. In fact, it turns out to be question that is generally not easy to resolve.

Let's see how the graphs can be rebuilt going backwards starting at $G_7 = [0,0], G_6 = [1,1,0]$ and $G_5 = [1,1,1,1]$, which can only go one way:



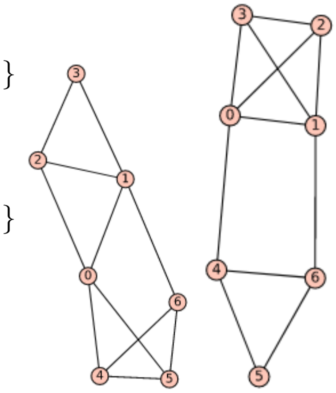
Then, as the diagram shows, $G_4, [2, 2, 2, 1, 1]$ can either be a connected graph or not: two distinct types for sure. This leads to even more possibilities for $G_3, [3, 3, 2, 2, 2, 2]$, the first of which I used the Sagemath generator function to produce, but the others I defined by drawing and then producing the distinct dictionary for it.

```
ds=[3,3,2,2,2,2]
g = graphs.DegreeSequence(ds)
g.degree_sequence()
sage.graphs.graph_generators.graphs.DegreeSequence(ds)
g.plot()
#with dictionary, 1
gdict_04_1={0:[1,3,5],1:[2,3],2:[4],4:[5]}
g_04_1=Graph(gdict_04_1)
g_04_1.plot()
#with dictionary, 2
gdict_04_2={0:[1,2,5],1:[3,4],2:[4],3:[5]}
g_04_2=Graph(gdict_04_2)
g_04_2.plot()
#with dictionary, 3
gdict_04_3={0:[1,2,5],1:[3,4],2:[4],3:[5]}
g_04_3=Graph(gdict_04_3)
g_04_3.plot()
```

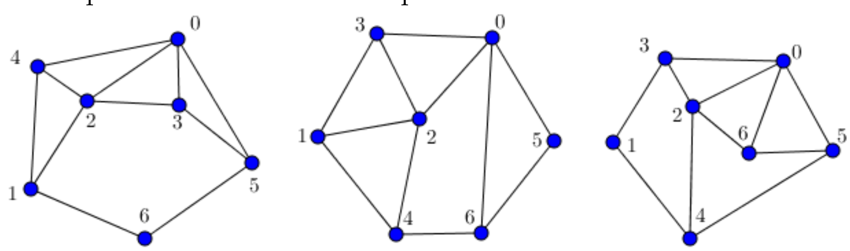


The first of these has these two sequels produced by this Sagemath code"

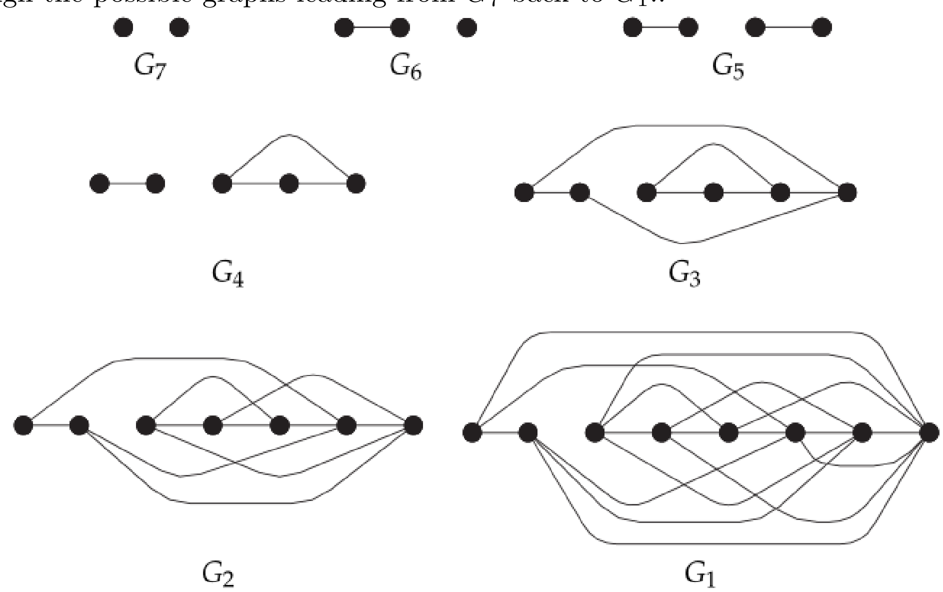
```
gdict_05_0={0:[1,2,4,5],1:[2,3,6],2:[3],4:[5,6],5:[6]}
g_05_0=Graph(gdict_05_0)
g_05_0.plot()
## next
gdict_05_1={0:[1,2,3,4],1:[2,3,6],2:[3],4:[5,6],5:[6]}
g_05_1=Graph(gdict_05_1)
g_05_1.plot()
```



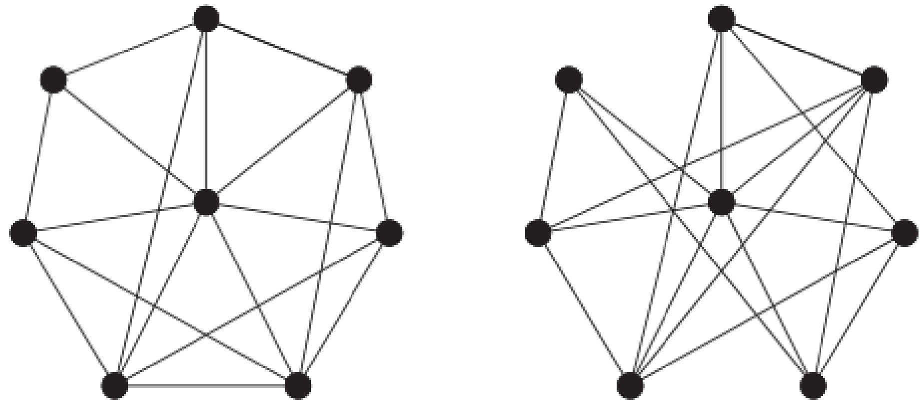
Then I used built these three sequels to the second [2, 2, 2, 1, 1] graphs, and copied them to Geogebra so I could make them planar and easier to compare:



We could continue on in this way to try to obtain an exhaustive lising of all non-isomorphic forms for each degree-sequence of our ladder, but suffice it to say that the sequence below is but one of many through the possible graphs leading from G_7 back to G_1::



Here, then, are two among an unknown number of different graphs with the degree sequence [7, 5, 5, 4, 4, 4, 4, 3]:



(d) Theorem:

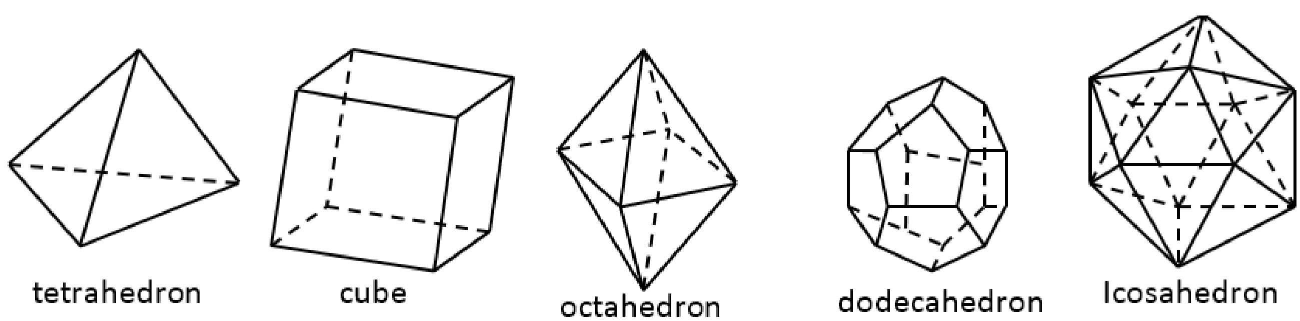
Consider a list $s = [d_1, d_2, \ldots, d_n]$ of n numbers in descending order. This list is *graphic* if and only if $s = [d_1, d_2, \ldots, d_{n-1}]$ of $n - 1$ numbers is graphic as well, where

$$d_i^* = \begin{cases} d_{i+1} - 1 & : \text{for } i = 1, 2, \ldots, d_1 \\ d_{i+1} & : \text{otherwise} \end{cases}$$

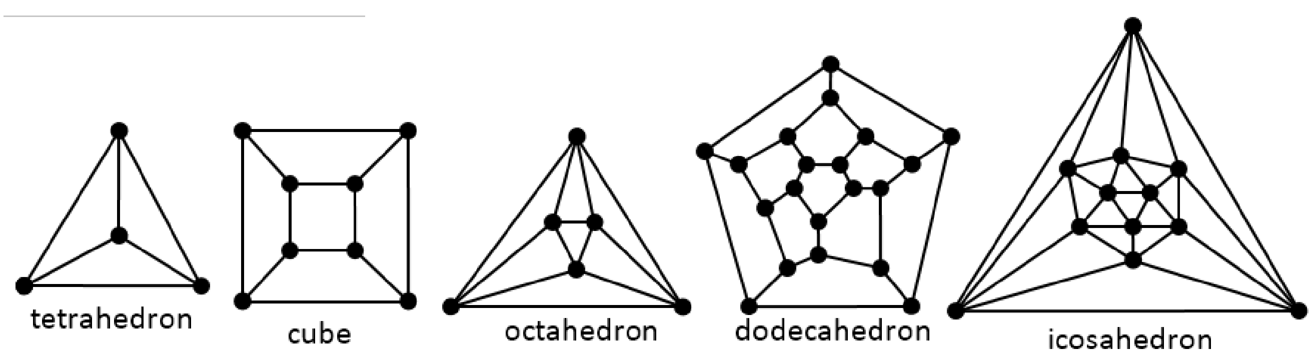
Apply this theorem to determine whether or not the degree list $[9, 6, 6, 4, 4, 4, 4, 3, 3, 2, 1, 1, 0]$ describes an actual graph.

ANS: $[9, 6, 6, 4, 4, 4, 4, 3, 3, 2, 1, 1, 0] \rightarrow [5, 5, 3, 3, 3, 3, 2, 2, 1, 1, 1, 0]$ One could continue to produce degree sequences according to the algorithm, but we've got a degree sequence here, the sum of whose degrees is odd, which violates Euler's theorem.

11. The following five solids are known as the Platonic Solids:

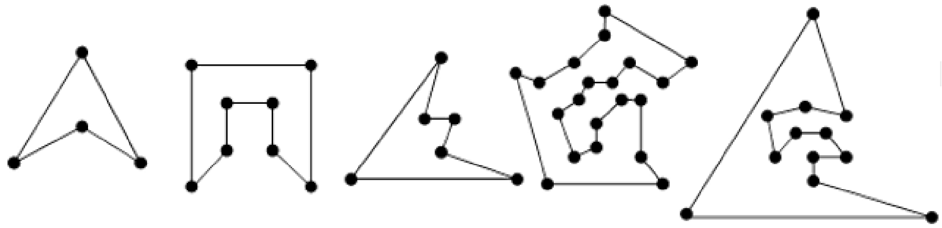


If you consider the edges and vertices of these solids as the edges and vertices of a graph, each can be represented in planar form:



(a) Which of the platonic graphs have an Euler circuit?
ANS: Only the octahedron has all even valences, thus it is the only one with an Euler circuit.

(b) Which of the platonic graphs have a Hamiltonian circuit?
ANS: They all have Hamiltonian circuits, as demonstrated by the subgraphs below.



(c) Euler's formula applies to polyhedra. It states that if n, m and f are the number of vertices, edges and faces, respectively, then $n - m + f = 2$. For example, for the tetrahedron, $4 - 6 + 4 = 2$. Verify Euler's formula for the other 4 platonic solids by writing out the equations that way.

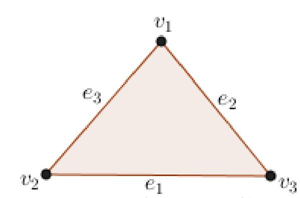
(d) The dual graph of a plane graph G is a graph that has a vertex for each face of G. The dual graph has an edge whenever two faces of G are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. Thus, each edge e of G has a corresponding dual edge, whose endpoints are the dual vertices corresponding to the faces on either side of e. Draw the dual graph for each of the platonic graphs. Are these also platonic?
ANS: We can tabulate the results as follows:

	Faces	Vertices	Edges	Formula
Tetrahedron	4 triangular	4	6	$4 + 4 - 6 = 2$
Hexahedron (cube)	6 square	8	12	$6 + 8 - 12 = 2$
Octahedron	8 triangular	6	12	$8 + 6 - 12 = 2$
Dodecahedron	12 pentagonal	20	30	$12 + 20 - 30 = 2$
Icosahedron	20 triangular	12	30	$20 + 12 - 30 = 2$

12. Let $X(G_0) = \{G | G \subseteq G_0\}$ be the set of all subgraphs of a graph G_0. Prove whether or not $(X(K_3), \text{subgraph})$

(a) is a poset.
ANS: We need to check *reflexivity*, *transitivity*, and *antisymmetry*. Referring to exercise 28 of §2.6, we define a subgraph to be a graph whose vertices and edges are all contained in the containing graph. Thus G is a subgraph of G_0 iff $V_G \subseteq V_{G_0}$ and $E_G \subseteq E_{G_0}$. Since every set is a subset of itself, every graph is a subgraph of itself, and the *reflexivity* requirement is met. Similarly, the subgraph relation inherits transitivity and antisymmetry from the subset relations transitivity and antisymmetry properties.

(b) is a lattice.

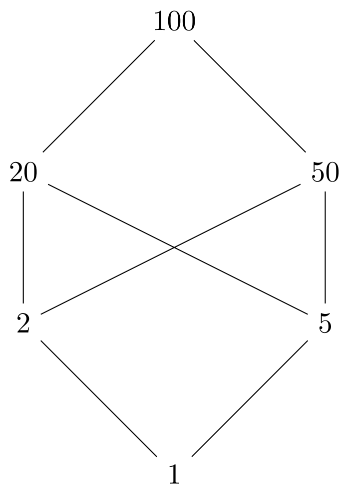


The question arises as to how to define the meet and join of two graphs to be consistent with the requirement that, for any subgraphs G_1 and G_2 of G_0 there is a largest $G \subseteq G = G_1 \wedge G_2$. Define a subgraph of K_3 by $Gx_1x_2x_3y_1y_2y_3$ where $x_i = 1$ iff vertex i is part of the the subgraph and $y_i = 1$ iff edge i is part of the subgraph. Thus $K_3 = G111111$ and $G000000 = \emptyset$. Note that the notation $G000001$ wouldn't make sense, since if $e_3 \in G_E$ then $v_1, v_2 \in G_V$. The 18 subgraphs are
$G111111, G111110, G111101, G111011, G111100, G111010, G111001, G110001, G101010,$ $G011100, G111000, G110000, G101000, G011000, G100000, G010000, G001000, G000000$. Then the meet of two graphs, $Gx_1x_2x_3y_1y_2y_3$ and $Gu_1u_2u_3v_1v_2v_3$ is $Ga_1a_2a_3b_1b_2b_3$ where $a_i = \min(x_i, u_i), b_i = \min(y_i, v_i)$. For the join of two graphs, $G_1 \vee G_2$ choose $a_i = \max(x_i, u_i), b_i = \max(y_i, v_i)$

(c) is a Boolean algebra.
ANS: Well, according to theorem 2.4, every Boolean algebra is isomorphic to some power set and so must have 2^n elements. Our set has 18, and so can't be a boolean algebra. Which of the distributivity, boundedness and complement properties are violated, and how?

13. Consider the divisibility poset, $(\{1, 2, 5, 20, 50, 100\}, |)$

(a) Draw the Hasse diagram of this poset.

ANS: 

(b) Determine whether or not this poset is a lattice.

ANS: Generally, with a division poset, we define the meet of two numbers is $a \wedge b = \texttt{gcd}(a, b)$ and the join is $a \vee b = \texttt{lcm}(a, b)$ Reflexivity holds since each number is divisible by itself. But what would $\texttt{gcd}(20, 50)$ be? 5? Both 2 and 5 are divisors of 20 and 50, and since 2 is less than 5, it makes sense to choose 5 as the gcd. So let's check the properties. Commutativity is not at issue, but what about associativity? $a \vee (b \vee c) = \texttt{lcm}(a, \texttt{lcm(b,c)})$ is good and so is $a \wedge (b \wedge c) = \texttt{gcd}(a, \texttt{gcd(b,c)})$. What about **absorption**? Is it true that $a \vee (a \wedge b) = \texttt{lcm}(a, \texttt{gcd(a,b)}) = a$? Let's try it with some numbers: $2 \vee (2 \wedge 5) = \texttt{lcm}(2, \texttt{gcd}(2, 5)) = \texttt{lcm}(2, 1) = 2$ and $2 \wedge (20 \vee 50) = \texttt{gcd}(2, \texttt{lcm}(20, 50)) = \texttt{gcd}(2, 100) = 2$. So, in genereal, if we take $a \wedge b = \max(x \texttt{ such that } x|a \wedge x|b)$ and $a \vee b = \min(x \texttt{ such that } b|x \texttt{ and } a|x)$ then we have the properties of a lattice satisfied.

14. Let Π_n be the poset of all set partitions of n. E.g., two elements of Π_5 are

$$S = \{1, 3, 4\}, \{2, 5\}$$

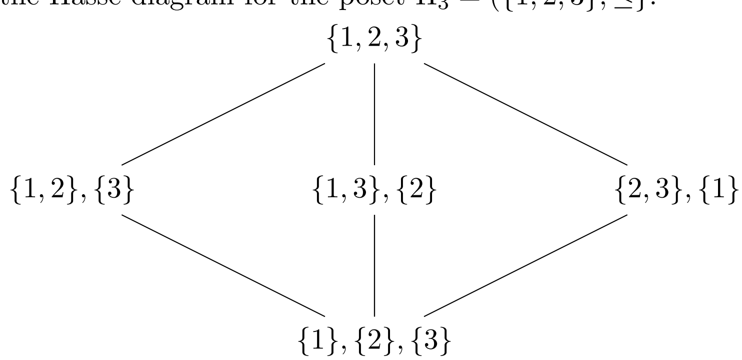
This is abbreviated 134|25

$$T = \{1, 3\}, \{4\}, \{2, 5\}$$

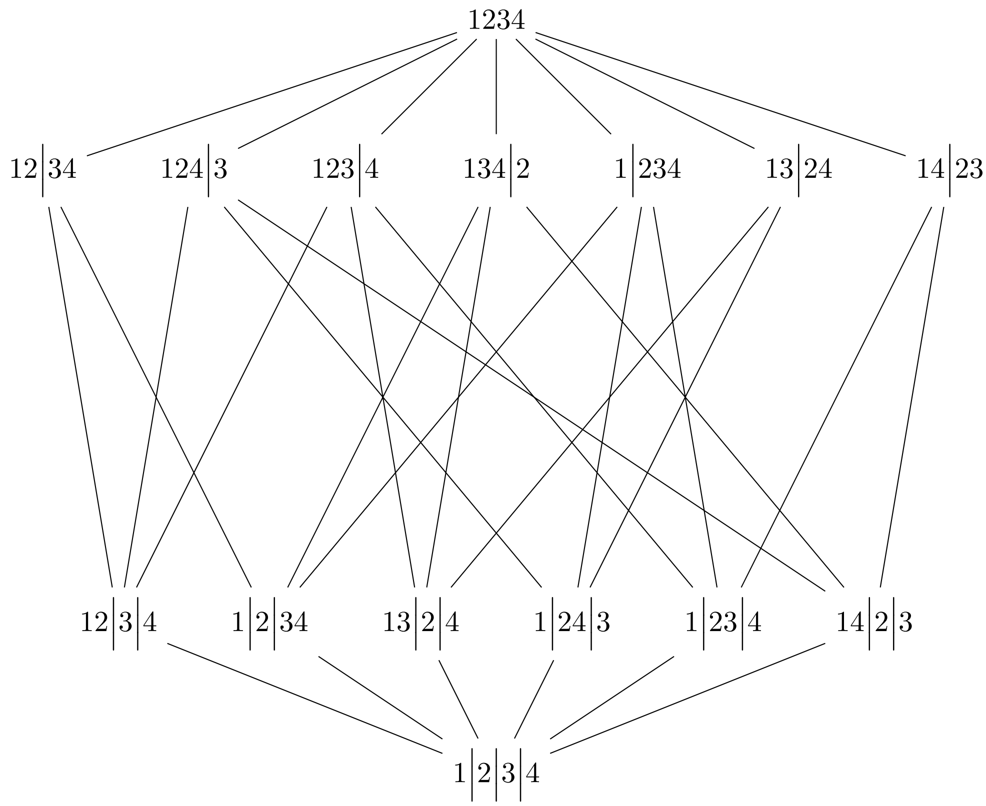
...abbreviated: $T = 13|4|25$

The sets $\{1, 3, 4\}$ and $\{2, 5\}$ are called the blocks of S. We can impose a partial order on Π_n by putting $T \leq S$ if every block of T is contained in a block of S; for short, T **refines** S.

(a) Draw the Hasse diagram for the poset $\Pi_3 = (\{1, 2, 3\}, \leq\}$.

ANS: 

(b) Draw the Hasse diagram for the poset $\Pi_4 = (\{1, 2, 3, 4\}, \leq\}$.



ANS:

(c) What are the subpartitions two levels down on the Hasse diagram for $\Pi_5 = (\{1,2,3,4,5\}, \leq)$?

ANS: It's instructive to do this by hand: start with the 15 ways to partition into two subsets:

$1|2345, 2|1345, 3|1245, 4|1235, 5|1234, 12|345, 13|245, 14|135, 15|234, 23|145, 24|135, 25|134, 34|125, 35|124, 45|123$

...and then do all the further partitions on each one, being careful not to duplicate.

Then there's the Sagemath command to list all the partitions:

```
SetPartitions(5).list()
#
[{{1, 2, 3, 4, 5}}, {{1}, {2, 3, 4, 5}}, {{1, 3, 4, 5}, {2}},
{{1, 2, 4, 5}, {3}}, {{1, 2, 3, 5}, {4}}, {{1, 2, 3, 4}, {5}},
{{1, 2}, {3, 4, 5}}, {{1, 3}, {2, 4, 5}}, {{1, 4}, {2, 3, 5}},
{{1, 5}, {2, 3, 4}}, {{1, 4, 5}, {2, 3}}, {{1, 3, 5}, {2, 4}},
{{1, 3, 4}, {2, 5}}, {{1, 2, 5}, {3, 4}}, {{1, 2, 4}, {3, 5}},
{{1, 2, 3}, {4, 5}}, {{1}, {2}, {3, 4, 5}}, {{1}, {2, 4, 5}, {3}},
{{1}, {2, 3, 5}, {4}}, {{1}, {2, 3, 4}, {5}}, {{1, 4, 5}, {2}, {3}},
{{1, 3, 5}, {2}, {4}}, {{1, 3, 4}, {2}, {5}}, {{1, 2, 5}, {3}, {4}},
{{1, 2, 4}, {3}, {5}}, {{1, 2, 3}, {4}, {5}}, {{1}, {2, 3}, {4, 5}},
{{1}, {2, 4}, {3, 5}}, {{1}, {2, 5}, {3, 4}}, {{1, 3}, {2}, {4, 5}},
{{1, 4}, {2}, {3, 5}}, {{1, 5}, {2}, {3, 4}}, {{1, 2}, {3}, {4, 5}},
{{1, 4}, {2, 5}, {3}}, {{1, 5}, {2, 4}, {3}}, {{1, 2}, {3, 5}, {4}},
{{1, 3}, {2, 5}, {4}}, {{1, 5}, {2, 3}, {4}}, {{1, 2}, {3, 4}, {5}},
{{1, 3}, {2, 4}, {5}}, {{1, 4}, {2, 3}, {5}}, {{1}, {2}, {3}, {4, 5}},
{{1}, {2}, {3, 5}, {4}}, {{1}, {2}, {3, 4}, {5}},
{{1}, {2, 5}, {3}, {4}}, {{1}, {2, 4}, {3}, {5}},
{{1}, {2, 3}, {4}, {5}}, {{1, 5}, {2}, {3}, {4}},
{{1, 4}, {2}, {3}, {5}}, {{1, 3}, {2}, {4}, {5}},
{{1, 2}, {3}, {4}, {5}}, {{1}, {2}, {3}, {4}, {5}}]
```

...from which we can pick out the partitions into 3:

```
{{1}, {2}, {3, 4, 5}}, {{1}, {2, 4, 5}, {3}}, {{1}, {2, 3, 5}, {4}},
{{1}, {2, 3, 4}, {5}}, {{1, 4, 5}, {2}, {3}}, {{1, 3, 5}, {2}, {4}},
{{1, 3, 4}, {2}, {5}}, {{1, 2, 5}, {3}, {4}}, {{1, 2, 4}, {3}, {5}},
{{1, 2, 3}, {4}, {5}}, {{1}, {2, 3}, {4, 5}}, {{1}, {2, 4}, {3, 5}},
{{1}, {2, 5}, {3, 4}}, {{1, 3}, {2}, {4, 5}}, {{1, 4}, {2}, {3, 5}},
{{1, 5}, {2}, {3, 4}}, {{1, 2}, {3}, {4, 5}}, {{1, 4}, {2, 5}, {3}},
{{1, 5}, {2, 4}, {3}}, {{1, 2}, {3, 5}, {4}}, {{1, 3}, {2, 5}, {4}},
{{1, 5}, {2, 3}, {4}}, {{1, 2}, {3, 4}, {5}}, {{1, 3}, {2, 4}, {5}},
{{1, 4}, {2, 3}, {5}}
```

...or, using our notation:

$1|2|345$, $1|3|245$, $1|4|235$,

$1|5|234$, $2|3|145$, $2|4|135$,

$2|5|134$, $3|4|125$, $3|5|124$,

$4|5|123$, $1|23|45$, $1|24|35$,

$1|25|34$, $13|2|45$, $14|2|35$,

$15|2|34$, $12|3|45$, $14|3|25$,

$15|3|24$, $12|4|35$, $13|4|25$,

$15|4|23$, $12|5|34$, $13|5|24$,

$14|23|5$

15. Pose an interesting question regarding the substance of chapters 1 and 2.

- Can every simple connected graph on n vertices can be decomposed into at most $\frac{1}{2}(n+1)$ paths?
- How many non-isomorphic graphs can be constructed with a given degree sequence?