Math 15 - Spring '17 – Chapters 1 and 2 Test Show detailed work for credit. Don't copy other people's work. Due Monday, 3/26/12

1. Consider the declaratives statements,

   P: The moon shares nothing. Q: It is the sun that shares our works. R: The earth is alive with creeping men.

   Using logical connectives, write a proposition which symbolizes the following:

   (a) If the moon shares nothing and it is the sun that shares our works, then the earth is alive with creeping men.

   (b) The earth is alive with creeping men only if the moon shares nothing.

   (c) If the moon shares everything or it is the sun does not share our works then the the earth is not alive with creeping men.

   (d) If the moon shares nothing and the sun does not share our works, then the earth is alive with creeping men.

2. Determine which of the following statements is true. _ Exactly one of these statements is false.
   _ Exactly two of these statements are false.
   _ Exactly three of these statements are false.
   _ Exactly four of these statements are false.
   _ Exactly five of these statements are false.
   _ Exactly six of these statements are false.
   _ Exactly seven of these statements are false.
   _ Exactly eight of these statements are false.
   _ Exactly nine of these statements are false.
   _ Exactly ten of these statements are false.

3. Construct a truth table for the proposition $P = [p \to (q \vee r)] \wedge \neg(p \leftrightarrow \neg r)]$.

4. Suppose we know A is true, $A \to (B \to C)$ is true and that $B \to D$ is true. Can we conclude that $\neg C \to D$? Prove your claim.

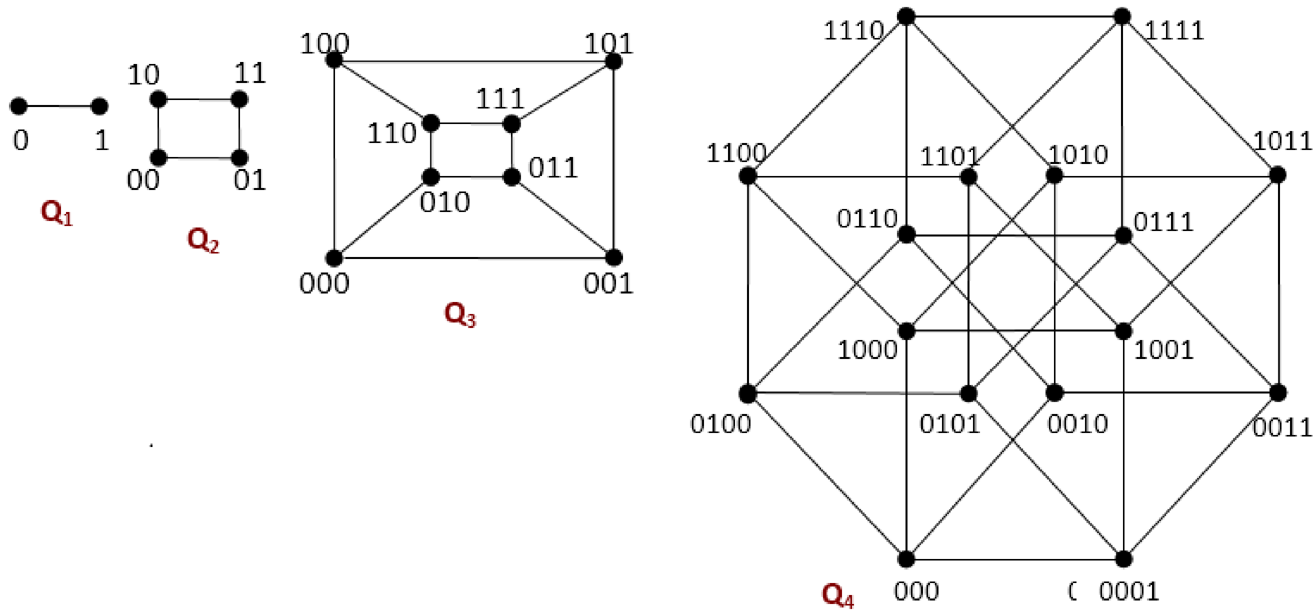
5. Given that $\neg A \to B$ and $B \to A$ and $A \to \neg B$ can we conclude $A \wedge \neg B$? Use the method of indirect proof to prove or disprove.

6. Prove that implication is not associative.

7. Prove or disprove: $(A \to B) \vee (A \to C) \to B \vee C$ is equivalent to $(A \vee B \vee C)$.

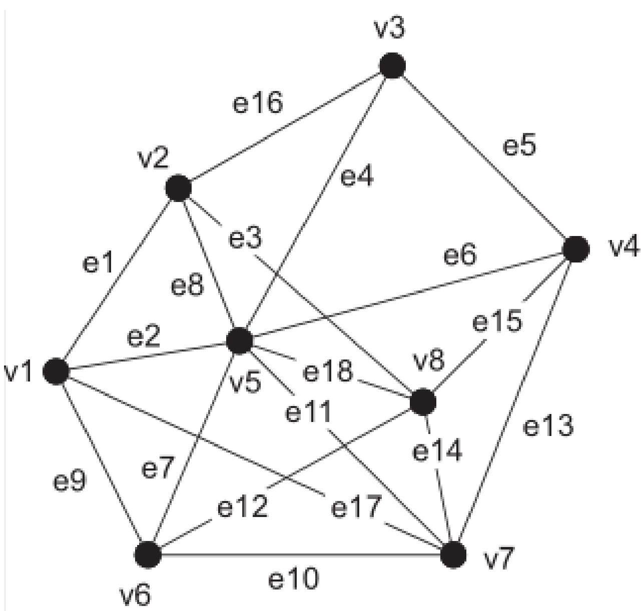
8. First, translate the following argument into propositional logic. Then, prove that the argument is valid using the method of formal derivation. Explain your answer.

   Sally is destined to be either a fearless adventurer or a great psychiatrist. If Nabokov is not a great writer then Sally is not destined to be a great psychiatrist. Yet Sally is clearly not destined to be fearless adventurer. So Nabokov is definitely a great writer.



9. Of particular importance in coding theory are the hypercube graphs, which may be constructed by taking as vertices all binary words (sequences of 0s and 1s) of a given length and joining two of these vertices if the corresponding binary words differ in just one place. The graph obtained in this way from the binary words of length k is called the k-hypercube (or k-dimensional cube), and is denoted Q_k. Cube graphs for $k = 1, 2, 3$ and 4 are shown above. Give a formula for the number of edges of a Q_k graph.

10. Listing the vertex degrees of a graph gives us a **degree sequence**. The vertex degrees are usually listed in descending order, in which case we refer to an ordered degree sequence. For example, if we consider the eight vertices of graph G below,



$V(G) = \{v_1, \dots, v_8\}$

$E(G) = \{e_1, \dots, e_{18}\}$

$e_1 = \langle v_1, v_2 \rangle$	$e_{10} = \langle v_6, v_7 \rangle$
$e_2 = \langle v_1, v_5 \rangle$	$e_{11} = \langle v_5, v_7 \rangle$
$e_3 = \langle v_2, v_8 \rangle$	$e_{12} = \langle v_6, v_8 \rangle$
$e_4 = \langle v_3, v_5 \rangle$	$e_{13} = \langle v_4, v_7 \rangle$
$e_5 = \langle v_3, v_4 \rangle$	$e_{14} = \langle v_7, v_8 \rangle$
$e_6 = \langle v_4, v_5 \rangle$	$e_{15} = \langle v_4, v_8 \rangle$
$e_7 = \langle v_5, v_6 \rangle$	$e_{16} = \langle v_2, v_3 \rangle$
$e_8 = \langle v_2, v_5 \rangle$	$e_{17} = \langle v_1, v_7 \rangle$
$e_9 = \langle v_1, v_6 \rangle$	$e_{18} = \langle v_5, v_8 \rangle$

we have the following vertex degrees

vertex	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
degree	4	4	3	4	7	4	5	5

which, when ordering these degrees in descending order, leads to the ordered degree sequence

$$[7, 5, 5, 4, 4, 4, 4, 3]$$

If every vertex has the same degree, the graph is called regular. In a k-regular graph each vertex has degree k. As a special case, 3-regular graphs are also called **cubic graphs**.
If every vertex has the same degree, the graph is called regular. In a k-regular graph each vertex has degree k. As a special case, 3-regular graphs are also called **cubic graphs**.

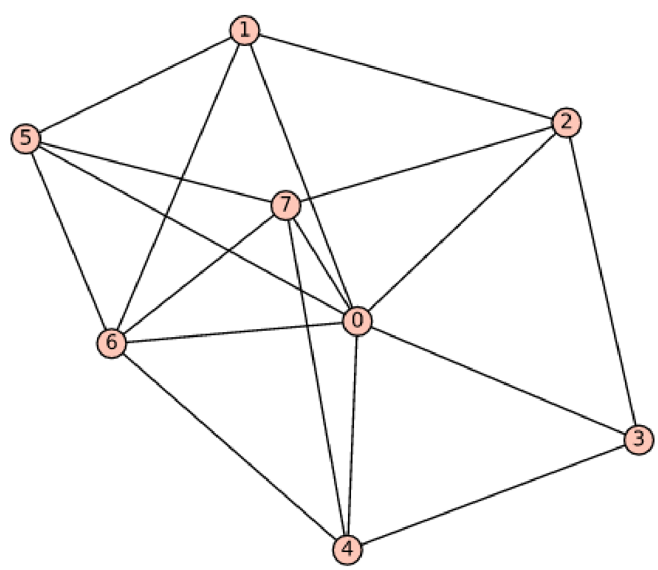
When considering degree sequences, it is common practice to focus only on simple graphs, that is, graphs without loops and multiple edges. An interesting question that comes to mind is when we are given a list of numbers, is there also a simple graph whose degree sequence corresponds to that list? There are some obvious cases where we already know that a given list cannot correspond to a degree sequence. For example, we have proven that the sum of vertex degrees is always even. Therefore, a minimal requirement is that the sum of the elements of that list should be even as well. Likewise, it is not difficult to see that, for example, the sequence $[4, 4, 3, 3]$ cannot correspond to a degree sequence. In this case, if this were a degree sequence, we would be dealing with a graph of four vertices. The first vertex is supposed to have four incident edges. In the case of simple graphs, each of these edges should be incident with a different vertex. However, there are only three vertices left to choose from, so $[4, 4, 3, 3]$ can never correspond to the degree sequence of a simple graph.

Does the degree list uniquely identify a graph? The list $[7, 5, 5, 4, 4, 4, 4, 3]$, for instance may not describe a unique graph, or any graph at all (if we pretend that we haven't already seen it for ourselves!) An algorithm for checking that a degree list describes an actual graph is to successively delete the highest degree vertex (and all its edges) from the graph and see if you maintain consistency with actual graphs.

In Sagemath, for instance, we can create the graph we started with (renaming the vertices) like so:

```
dg = {0: [1,2,3,4,5,6,7], 1: [2,5,6], 2: [3,7], 3: [4], 4: [6,7], \
5: [6,7], 6: [7] }
G1=Graph(dg)
G1.plot()
```

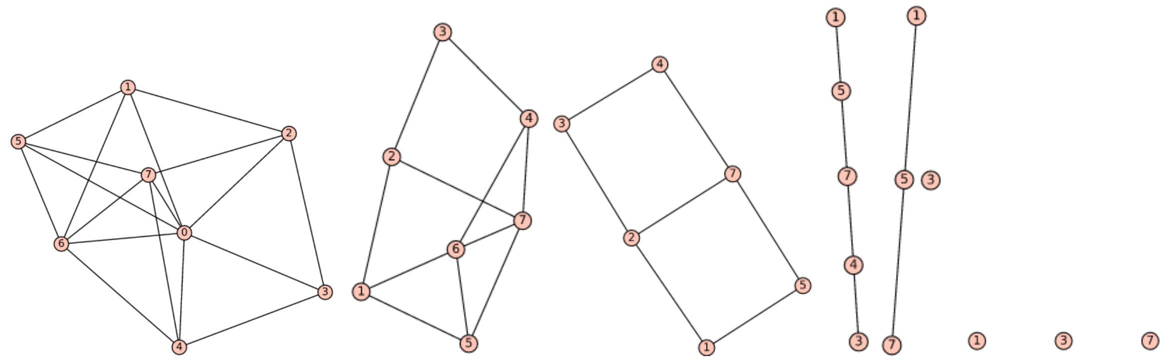
which produces this rendering:



Repeatedly applying the procedure

```
G1.delete_vertex(0);
G1.delete_vertex(6);
G1.delete_vertex(4);
G1.delete_vertex(2);
G1.delete_vertex(5);
```

produces a sequence of graphs:



(a) Will the procedure for verifying a degree list work if we delete vertices in any order whatsoever? If so, why choose to delete the highest degree vertices first?

(b) Construct two different graphs with the degree sequence $[3, 3, 2, 2, 2]$

(c) Construct two different graphs with the degree sequence $[7, 5, 5, 4, 4, 4, 4, 3]$

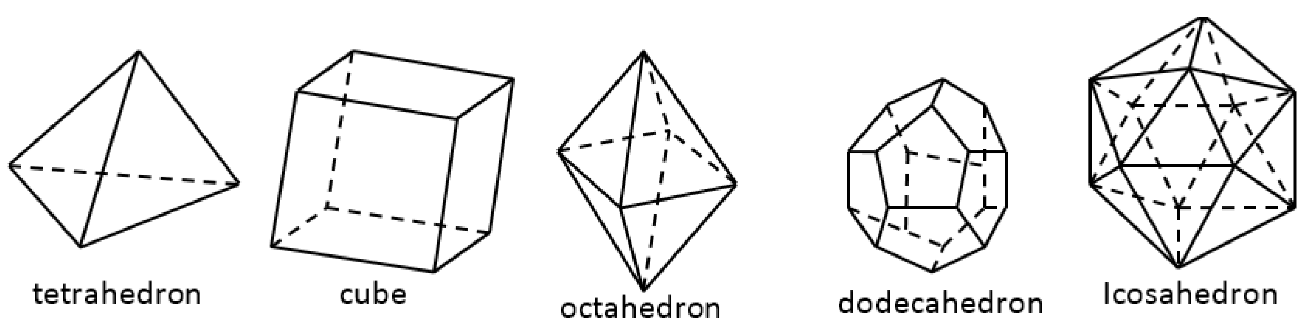
(d) Theorem:

Consider a list $s = [d_1, d_2, \ldots, d_n]$ of n numbers in descending order. This list is *graphic* if and only if $s = [d_1, d_2, \ldots, d_{n-1}]$ of $n-1$ numbers is graphic as well, where

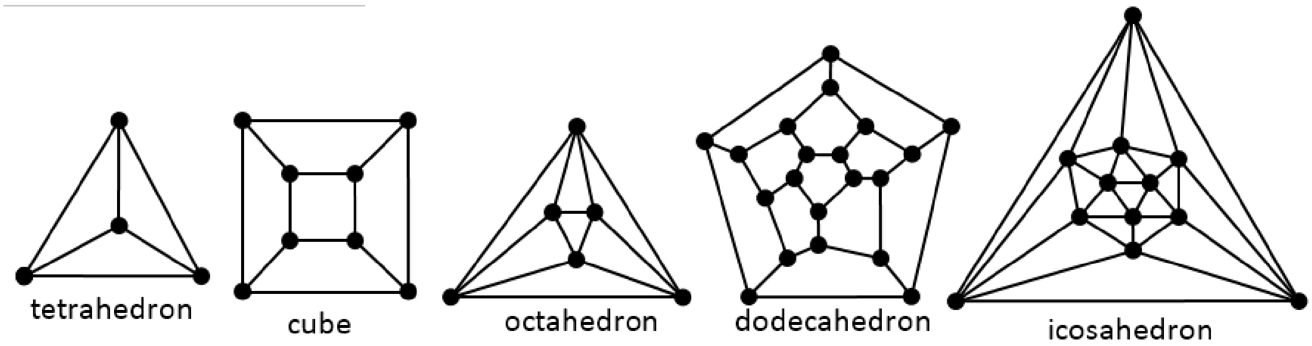
$$d_i^* = \begin{cases} d_{i+1} - 1 & : \text{for } i = 1, 2, \ldots, d_1 \\ d_{i+1} & : \text{otherwise} \end{cases}$$

Apply this theorem to determine whether or not the degree list $[9, 6, 6, 4, 4, 4, 4, 3, 3, 2, 1, 1, 0]$ describes an actual graph.

11. The following five solids are known as the Platonic Solids:



If you consider the edges and vertices of these solids as the edges and vertices of a graph, each can be represented in planar form:



(a) Which of the platonic graphs have an Euler circuit?

(b) Which of the platonic graphs have a Hamiltonian circuit?

(c) Euler's formula applies to polyhedra. It states that if n, m and f are the number of vertices, edges and faces, respectively, then $n - m + f = 2$. For example, for the tetrahedron, $4 - 6 + 4 = 2$. Verify Euler's formula for the other 4 platonic solids by writing out the equations that way.

(d) The dual graph of a plane graph G is a graph that has a vertex for each face of G. The dual graph has an edge whenever two faces of G are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. Thus, each edge e of G has a corresponding dual edge, whose endpoints are the dual vertices corresponding to the faces on either side of e. Draw the dual graph for each of the platonic graphs. Are these also platonic?

12. Let $X(G_0) = \{G | G \subseteq G_0\}$ be the set of all subgraphs of a graph G_0. Prove whether or not $(X(K_3), \text{subgraph})$

(a) is a poset.

(b) is a lattice.

(c) is a Boolean algebra.

13. Consider the divisibility poset, $(\{1, 2, 5, 20, 50, 100\}, |)$

(a) Draw the Hasse diagram of this poset.

(b) Determine whether or not this poset is a lattice.

14. Let Π_n be the poset of all set partitions of n. E.g., two elements of Π_5 are

$$S = \{1, 3, 4\}, \{2, 5\}\}$$

This is abbreviated 134|25

$$T = \{1, 3\}, \{4\}, \{2, 5\}$$

...abbreviated: $T = 13|4|25$
The sets $\{1, 3, 4\}$ and $\{2, 5\}$ are called the blocks of S. We can impose a partial order on Π_n by putting $T \leq S$ if every block of T is contained in a block of S; for short, T **refines** S.

(a) Draw the Hasse diagram for the poset $\Pi_3 = (\{1, 2, 3\}, \leq\}$.

(b) Draw the Hasse diagram for the poset $\Pi_4 = (\{1, 2, 3, 4\}, \leq\}$.

(c) What are the subpartitions two levels down on the Hasse diagram for $\Pi_5 = (\{1, 2, 3, 4, 5\}, \leq\}$?

15. Pose an interesting question regarding the substance of chapters 1 and 2.