CLASSROOM CAPSULES

Classroom Capsules are short (1–3 page) notes that contain new mathematical insights on a topic from undergraduate mathematics, preferably something that can be directly introduced into a college classroom as an effective teaching strategy or tool. Classroom Capsules should be prepared according to the guidelines on the inside front cover and submitted via Editorial Manager.

A Powerful Method of Non-Proof

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In the fall of 2013 I taught an introduction-to-abstract-mathematics course designed to help mathematics majors bridge the gap between their calculus courses and their upper-division proof-based courses. Shortly after introducing the students to truth tables, I ran across the following exercise from our text [**1**]: "Show that for any two statements ϕ and ψ either $\phi \Rightarrow \psi$ or its converse is true (or both)." This gave me pause. Although the claim could be verified through a truth table, it conflicted with my understanding that a conditional statement and its contrapositive could both be false. This provided a valuable opportunity for our class to explore truth tables and how they should be interpreted.

A previous exercise had instructed students to "use truth tables" to prove the equivalence of a conditional statement and its contrapositive. Table 1 legitimately demonstrates this equivalence, as can be seen from the tautology in the last column.

Table 1. A conditional statement and its contrapositive are equivalent.

ϕ	ψ	$\neg\phi$	$\neg\psi$	$\phi \Rightarrow \psi$	$\neg\psi \Rightarrow \neg\phi$	$(\phi \Rightarrow \psi) \Longleftrightarrow (\neg\psi \Rightarrow \neg\phi)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

The result is put to frequent use, as it is often more convenient to prove the contrapositive form rather than the original form of a statement. (For instance, supposing x to be an integer, try to directly prove the statement, "If x^2 is even, then x is even." Then try instead to prove its contrapositive, "If x is odd, then x^2 is odd.")

We can likewise use truth tables to prove that a conditional statement or its converse must be true; see Table 2.

In order to prove that a statement is true, it appears that we need only prove its converse to be false. Before scrutinizing this tool, we make use of it. Begin by taking on Fermat's last theorem: Let $n > 2$ and let x and y be positive integers.

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Table 2. Either a conditional statement or its converse must be true.

ϕ	ψ	$\phi \Rightarrow \psi$	$\phi \Leftarrow \psi$	$(\phi \Rightarrow \psi) \vee (\phi \Leftarrow \psi)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Fermat's last theorem (FLT). If z satisfies $z^n = x^n + y^n$, then z is not an integer.

Converse of FLT. If z is not an integer, then $z^n = x^n + y^n$.

"Proof" of FLT by falsity of the converse. Choose z to be any noninteger not equal to $\sqrt[n]{x^n + y^n}$. This establishes that the converse of FLT is false and thus, ostensibly, that FLT is true. ■

Finally, a short proof of FLT! However, this result has previously been established [**2**], so we should try something that is unknown. In the eighteenth century, Goldbach conjectured that every even number greater than 2 is the sum of two primes [**3**].

Goldbach's conjecture (GC). If x is an even number greater than 2, then there are primes p_1 and p_2 with $x = p_1 + p_2$.

Converse of GC. If p_1 and p_2 are primes and $x = p_1 + p_2$, then x is an even number greater than 2.

"Proof" of GC by falsity of the converse. Choose $p_1 = 2$ and $p_2 = 3$. Then $x = 5$, which is not an even number. ■

So far we have used our falsity-of-the-converse technique to "prove" one result that is already known to be true and another that is widely believed to be true. It may be difficult to find the fallacy of our argument from those examples. Thus we now attempt to prove something that is obviously false, that every even number is odd.

An even odder conjecture (EOC). If a number is even, then it is odd.

Converse of EOC. If a number is odd, then it is even.

"Proof" of EOC by falsity of the converse. The converse is clearly not true. ■

Okay, what went wrong? Notice that the antecedent of EOC, "a number is even," does not have a fixed truth value, and neither does the consequent, "it is odd." For instance, if we replaced "a number" by the specific number 3, then the converse of EOC reads, "If 3 is odd, then it is even." This is false and the original statement of EOC is vacuously true in this case. But if instead we replace "a number" by the specific number 2, then the converse reads, "If 2 is odd, then it is even." This is true and the original statement of EOC is false in this case.

How, then, should we interpret the motivating claim that either a conditional statement or its converse must be true? When the antecedent ϕ and the consequent ψ are both statements with fixed truth values, it is impossible for the statement $\phi \Rightarrow \psi$ and its converse $\phi \Leftarrow \psi$ to both be false; we can legitimately infer this from the truth tables. But if the antecedent or the consequent has variable truth values, then perhaps, as in EOC, the converse is false in some cases but true in other cases. In such situations, the general converse would be considered false but the original statement might not be true, because there might be individual cases in which the converse is true but the original statement is false.

I assigned my students the problem of creating a false conditional statement whose converse is also false (most of them were able to accomplish this) and of explaining why this is consistent with Table 2 (a few of them did a good job of this). I wish I had also offered them the opportunity to identify specifically where our "proofs" of Fermat's last theorem and Goldbach's conjecture failed—but I had not yet thought of those proofs. I invite the reader to scrutinize those arguments at this time.

Summary. Although truth tables can be used in a legitimate way to justify arguments, one should exercise caution when doing so. We demonstrate by suggesting a method of proof that is too good to be true.

References

1. K. Devlin, *Introduction to Mathematical Thinking*. Keith Devlin, Palo Alto, CA, 2012.
2. I. Kleiner, From Fermat to Wiles: Fermat's last theorem becomes a theorem, *Elem. Math.* **55** (2000) 19–37, http://dx.doi.org/10.1007/PL00000079.
3. D. Wells, *Prime Numbers: The Most Mysterious Figures in Math*. Wiley, Hoboken, NJ, 2005.

