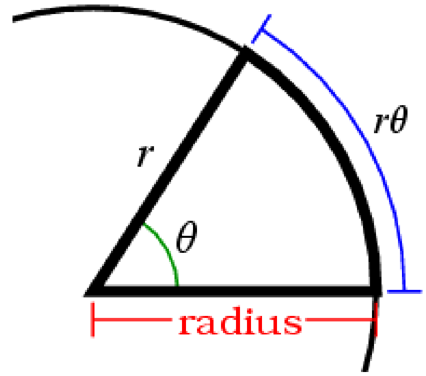
Math 5 – Trigonometry – Review Sheet for Chapter 5

Key Ideas:

Def: Radian measure of an angle is the ratio of arclength subtended by that central angle to the radius of the circle: $\theta \equiv \frac{s}{r} \Rightarrow s = r\theta$



$180^\circ = \pi$ radians.

The factor $\frac{\pi}{180^\circ}$ converts from degrees to radians.

The factor $\frac{180^\circ}{\pi}$ converts from radians to degrees.

In a circle of radius r the area A of a sector with central angle θ (measured in radians) satisfies the proportional relation $\frac{\theta}{2\pi} = \frac{A}{\pi r^2}$, so $A = \frac{\theta r^2}{2}$.

Angular speed is measured in radians per unit time: $\omega = \frac{\Delta\theta}{\Delta t}$. It can also be measured in rotations per second (rpm.)

The linear speed of a point on the perimeter of a rotating disk of radius r with angular speed ω is $v = \frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$. Note that this means $v = \omega r$.

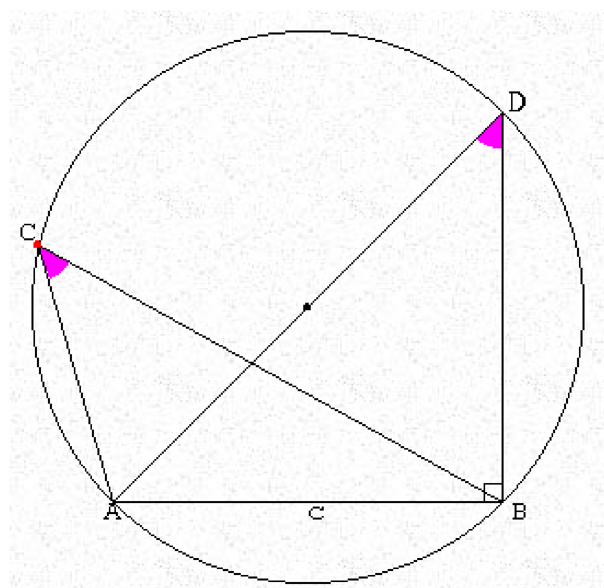
The area of a triangle with angle θ between sides of lengths a and b is given by $A = \frac{1}{2}ab\sin\theta$.

The law of sines says that the diameter of the circumscribing circle for any triangle is expressible as $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$

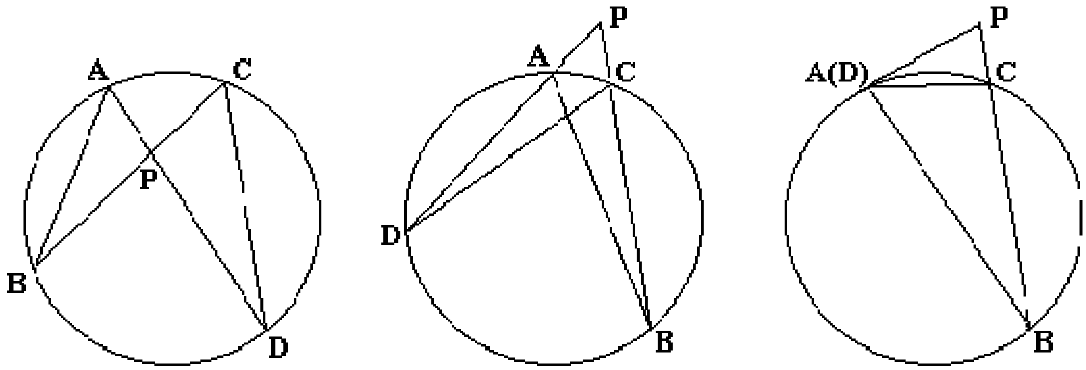
For example, in triangle ABC, we can move point C along the perimeter of the circumscribing circle until it's opposite point A. Then angle B is 90 degrees so that $\sin C = \sin D = \frac{c}{\text{diameter}}$.

This is equivalent to the diameter = $\frac{c}{\sin C}$

Similarly diameter = $\frac{b}{\sin B} = \frac{a}{\sin A}$.



Power of a Point Theorem



Given a point P, a circle and two lines through P intersecting the circle in points A and D and, respectively, B and C. Then $AP \cdot DP = BP \cdot CP$. The point P may lie either inside or outside the circle. The line through A and D (or that through B and C or both) may be tangent to the circle, in which case A and D coalesce into a single point. In all the cases, the theorem holds and is known as the **Power of a Point Theorem**.

When the point P is inside the circle, the theorem is also known as the **Theorem of Intersecting Chords** (or the **Intersecting Chords Theorem**) and has a beautiful interpretation. When the point P is outside the circle, the theorem becomes the **Theorem of Intersecting Secants** (or the **Intersecting Secants Theorem**.)

The proof is exactly the same in all three cases mentioned above. Since triangles ABP and CDP are similar, the following equality holds:

$$AP/CP = BP/DP,$$

which is equivalent to the statement of the theorem: $AP \cdot DP = BP \cdot CP$.

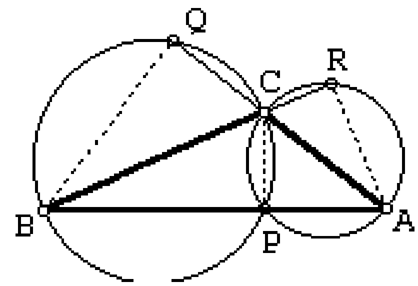
The common value of the products then depends only on P and the circle and is known as the *Power of Point P with Respect to the Circle*. Note that, when P lies outside the circle, its power equals the length of the square of the tangent from P to the circle.

The theorem is reversible: Assume points A, B, C, and D are not collinear. Let P be the intersection of AD and BC such that $AP \cdot DP = BP \cdot CP$. Then the four points A, B, C, and D are concyclic. To see that draw a circle through, say, A, B, and C. Assume it intersects AP at D'. Then, as was shown above, $AP \cdot D'P = BP \cdot CP$, from which $D = D'$. (If, say, B and C coincide, draw the circle through A tangent to PB at B.)

The Law of Cosines

For a triangle with sides a, b, and c and angle θ opposite the side c, one has

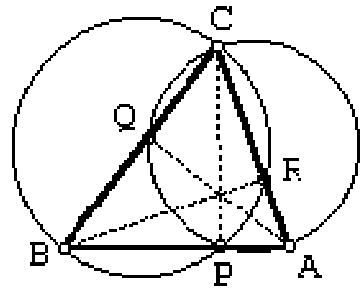
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



<u>Proof</u>

Let a and b be lengths of sides BC and AC, respectively, and consider the length, c, of side AB, as a function of θ, the magnitude of the angle at C.

There are three cases:



If the triangle is acute, construct the three altitudes, so that the circles with diameters BC and AC will intersect the circle at the feet of the altitudes, R, P and Q.

Why do these circles intersect the altitude feet? Because the diameters form a straight angle and there is a theorem that proves that an arc of the circle subtends an angle at the opposite circumference which is half the angle subtended at the center of the circle. Half a straight angle is a right angle. [As is shown in one of the proofs of Pythagoras' theorem], the foot of the altitude from C to AB and the two circles coincide at a point, say P, which cuts AB into segments BP and PA of lengths x and y, respectively. Q, the foot of the altitude from A, is the right angle of right triangle AQC, and Q lies on the circle with diameter AC. Inspection confirms that QC is of length $b\cos(\theta)$. Similarly, R, the foot of the altitude from B, R lies on circle BC, and RC is of length $a\cos(\theta)$. Then by the Power Theorem, the power of the point B with respect to circle AC, is $a(a - b\cos(\theta)) = xc$. Similarly, for the power of the point A with respect to circle BC, we have $b(b - a\cos(\theta)) = yc$. Adding the two equations yields $a^2 + b^2 - 2ab\cos(\theta) = (x + y)c = c^2$, which is the Law of Cosines.

If the original triangle is obtuse, we have two further cases:

If angle C is obtuse, then the altitude from C cuts side AB say, at P, but the altitudes from A and B lie outside the triangle. The feet of these altitudes are found at the intersection of sides BC and AC, produced, with the circles with diameters AC and BC, respectively; call them Q and R. As before, let x and y denote the lengths of BP and PA. Since angle C is obtuse, $\cos(\theta)$ is less than zero, and the lengths of RC and QC are and $-a\cos(\theta)$ and $-b\cos(\theta)$, respectively. Now the power of B with respect to circle AC is
$a(a - b\cos(\theta)) = xc$, and the power of A with respect to circle BC is
$b(b - a\cos(\theta)) = yc$. As before, adding the two equations yields the Law of Cosines.

Finally, if angle C is acute, but, say, angle B is obtuse, then the altitude from B cuts side AC, say at R, but the altitudes from A and C lie outside the triangle, and meet the sides CB and AB, produced, respectively, at, say Q and P. As before, Q and P lie on the circles with diameters AC and BC, respectively. Denote the distance BP by z. Since the length of RC is $a\cos(\theta)$, while the length of QC is $b\cos(\theta)$, the power of A with respect to circle BC is $b(b - a\cos(\theta)) = (c + z)c$, while, using the Power Theorem for an interior point, we have for the power of B with respect to circle AC is
$a(b\cos(\theta) - a) = zc$. In this case, *subtracting* the second equation from the first yields the Law of Cosines. QED.

General Strategies for Solving Triangles

If you don't have a right triangle then the given information falls into one of the following types:

- **SSS**: If the three sides of a triangle are known, first use the Law of Cosines to find one of the angles. It is usually best to find the largest angle first, the one opposite the longest side. Then, set up a proportion using the Law of Sines to find the second angle. Finally, subtract these angle measures from 180° to find the third angle.
- **SAS:** If two sides and the included angle of a triangle are known, first use the Law of Cosines to solve for the third side. Next, use the Law of Sines to find the smaller of the two remaining angles. This is the angle opposite the shortest or shorter side, not the longest side. Finally, subtract these angle measures from 180° to find the third angle. You can use the Law of Cosines to find the two missing angles, but using the Law of Cosines is usually more complex than the Law of Sines.
- **ASA:** If two angles and the included side of a triangle are known, first subtract these angle measures from 180° to find the third angle. Next, use the Law of Sines to set up proportions to find the lengths of the two missing sides. You could use the Law of Cosines to find the length of the third side, but why bother if you can use the simpler Law of Sines instead?
- **AAS:** If two angles and a non-included side of a triangle are known, first subtract these angle measures from 180° to find the third angle, then you can treat it like an ASA situation.
- **ASS:** This is known as *the ambiguous case*. If two sides and a non-included angle of a triangle are known, there are *six* possible configurations, two if the given angle is obtuse or right and four if the given angle is acute. These six possibilities are shown in Figures 1 2 and 3 . In Figures 1 and 2 , h is an altitude where $h = a \sin \beta$ and β is an acute angle.

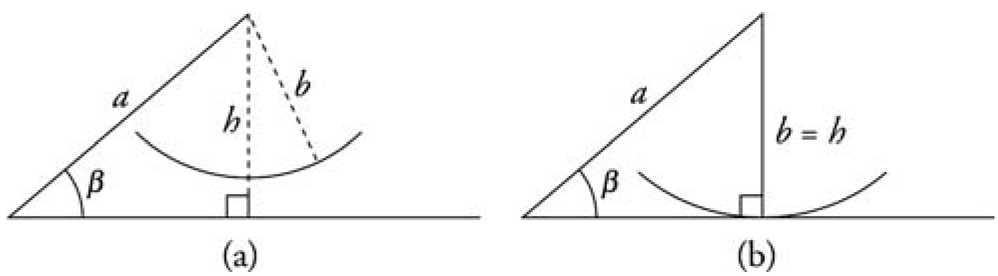


Figure 1 Two cases for SSA.

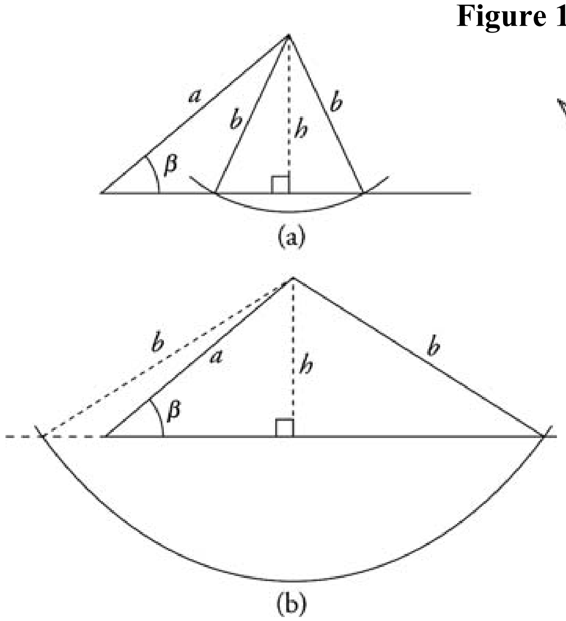


Figure 2 Ambiguous cases for SSA.

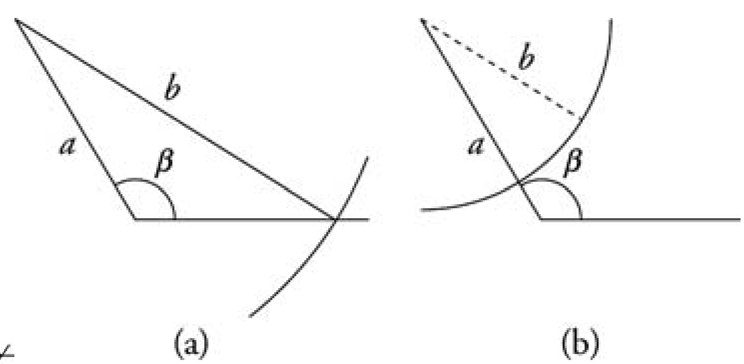


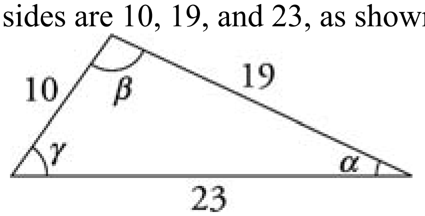
Figure 3 Two cases for SSA

- In Figure 1 (a), if $b < h$, then b cannot reach the other side of the triangle, and no solution is possible. This occurs when $b < a \sin\beta$.

- In Figure 1 (b), if $b = h = a \sin \beta$, then exactly one right triangle is formed.

- In Figure 2 (a), if $h < b < a$—that is, $a \sin \beta < b, < a$—then two different solutions exist.
- In Figure 2 (b), if $b = a$, then only one solution exists, and if $b = a$, then the solution is an isosceles triangle.
- If β is an obtuse or right angle, the following two possibilities exist.
- In Figure 3 (a), if $b > a$, then one solution is possible.
- In Figure 3 (b), if $b \leq a$, then no solutions are possible.

Example 1: (SSS) Find the difference between the largest and smallest angles of a triangle if the lengths of the sides are 10, 19, and 23, as shown.



First, use the Law of Cosines to find the size of the largest angle (β) which is opposite the longest side. Next, use the Law of Sines to find the size of the smallest angle (α), which is opposite the shortest side (10).

Thus, the difference between the largest and smallest angle is

$$100.3° - 25.33° = 74.97°$$

$$23^2 = 10^2 + 19^2 - (2)(10)(19)\cos\beta$$

$$\cos\beta = \frac{10^2 + 19^2 - 23^2}{(2)(10)(19)}$$

$$\cos\beta = \frac{-68}{380}$$

$$\cos\beta = -0.1789$$

$$\beta \approx Cos^{-1} - 0.1789$$

$$\beta \approx 100.3°$$

$$\frac{\sin\alpha}{10} \approx \frac{\sin 100.3°}{23}$$

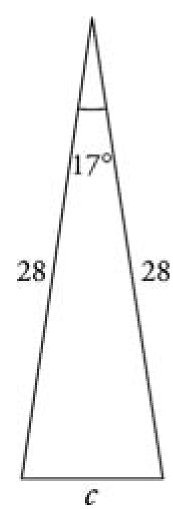
$$\sin\alpha \approx \frac{(10)(0.9839)}{23}$$

$$\sin\alpha \approx 0.4278$$

$$\alpha \approx Sin^{-1} - 0.4278$$

$$\alpha \approx 25.33°$$

Example 2: (SAS) The legs of an isosceles triangle have a length of 28 and form a 17° angle. What is the length of the third side of the triangle?



This is a direct application of the Law of Cosines.

$$c^2 = 28^2 + 28^2 - (2)(28)(28)\cos 17°$$

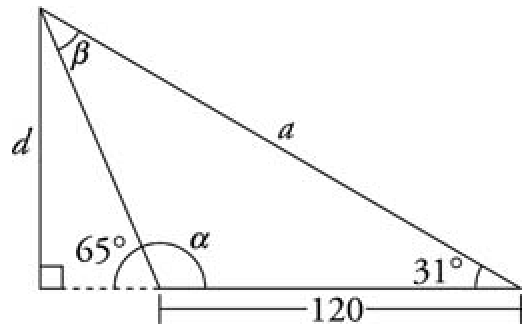
$$c^2 \approx 784 - 784 - (1568)(0.9563)$$

$$c^2 \approx 68.52$$

$$c \approx \sqrt{68.52}$$

$$c \approx 8.278$$

Example 3: (ASA) Find the value of d as shown.



First, calculate the sizes of angles α and β. Then find the value of a using the Law of Sines. Finally, use the definition of the sine to find the value of d.

$$\alpha = 180° - 65° = 115°$$

$$\beta = 180° - 115° - 31° = 34°$$

$$\frac{a}{\sin 115°} = \frac{120}{\sin 34°}$$

$$a = \frac{(120)(\sin 115°)}{\sin 34°}$$

$$a \approx \frac{(120)(0.9063)}{0.5592}$$

$$a \approx 194.5$$

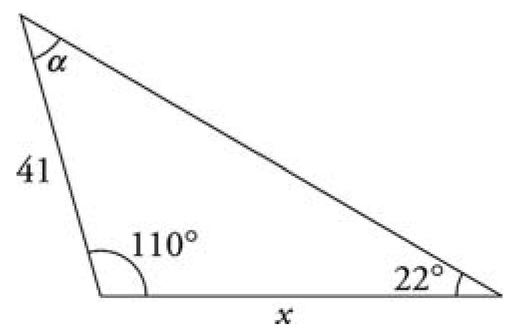
Finally

$$\sin 31° \approx \frac{d}{194.5}$$

$$d \approx (194.5)(0.5150)$$

$$d \approx 100.2$$

Example 4: (AAS) Find the value of x in the figure at right.



First, calculate the size of angle α. Then use the Law of Sines to calculate the value of x.

$$\alpha = 180° - 110° - 22° = 48°$$

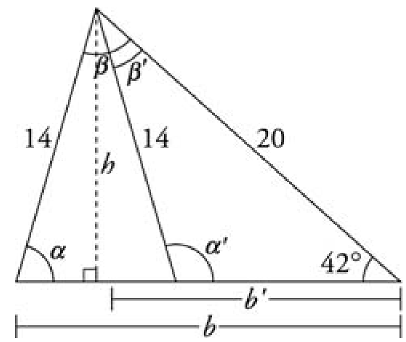
$$\frac{x}{\sin 48°} = \frac{41}{\sin 22°}$$

$$x \approx \frac{(41)(\sin 48°)}{\sin 22°}$$

$$x \approx \frac{(41)(0.7431)}{0.3746}$$

$$x \approx 81.33$$

Example 5: (SSA) One side of a triangle, of length 20, forms a 42° angle with a second side of the triangle. The length of the third side of the triangle is 14. Find the length of the second side.



The length of the altitude (h) is calculated first so that the number of solutions (0, 1, or 2) can be determined.

$$h = (20)\left(\sin 42^\circ\right) \approx (20)(0.6691) \approx 13.38$$

Because 13.38 < 14 < 20, there are two distinct solutions.

Solution 1: Use of the Law of Sines to calculate α.

$$\frac{\sin\alpha}{20} = \frac{\sin 42^\circ}{14}$$

$$\sin\alpha = \frac{(20)\left(\sin 42^\circ\right)}{14}$$

$$\sin\alpha = \frac{(20)(0.6691)}{14}$$

$$\sin\alpha \approx 0.9559$$

$$\alpha \approx Sin^{-1} 0.9559$$

$$\alpha \approx 72.92^\circ$$

Use the fact that there are 180° in a triangle to calculate β

$$\beta = 180^\circ - 42^\circ - \alpha$$

$$\beta = 180^\circ - 42^\circ - 72.92^\circ$$

$$\beta = 72.92^\circ$$

Use the Law of Sines to find the value of b.

$$\frac{b}{\sin\beta} = \frac{14}{\sin 42^\circ}$$

$$b = \frac{(14)\left(\sin\beta\right)}{\sin 42^\circ}$$

$$b \approx \frac{(14)(0.9069)}{0.6691}$$

$$b \approx 18.98$$

Solution 2: Use α to find α', and α' to find β'

$$\alpha' = 180^\circ - \alpha \approx 180^\circ - 72.92^\circ \approx 107.08^\circ$$

$$\beta' = 180^\circ - 42^\circ - \alpha' \approx 180^\circ - 42^\circ - 107.08^\circ \approx 30.92^\circ$$

Next, use the Law of Sines to find b'.

$$\frac{b'}{\sin\beta'} = \frac{14}{\sin 42^\circ}$$

$$b' = \frac{(14)\left(\sin\beta'\right)}{\sin 42^\circ}$$

$$b' = \frac{(14)(0.5138)}{0.6691}$$

$$b' \approx 10.75$$

Math 5 – Trigonometry – Chapter 5 Review Problems.

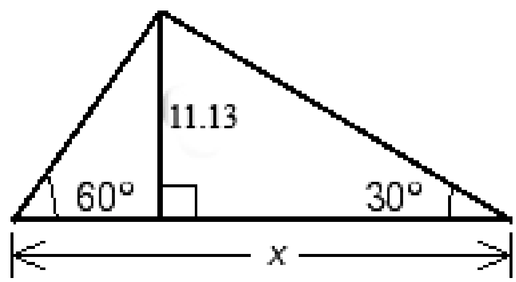
1. What is the length of the arc subtended by an angle $\theta = \frac{\pi}{7}$ in a circle of radius 14?

2. Find the central angle on a circle of radius 4 that subtends and arc of length 6. Give both the radian measure and the degree measure of the angle.

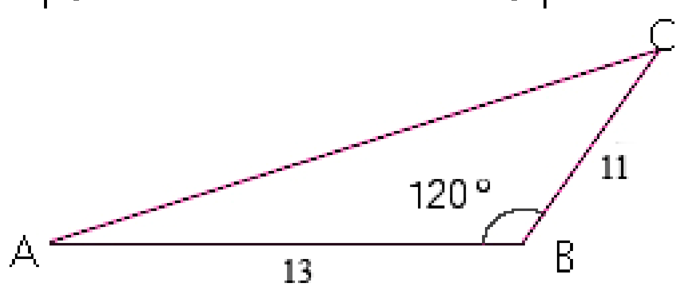
3. What is the radius of a circle where a sector with central angle of 36° has area = 9?

4. The angle of elevation to the top of the Diesel Mechanics building from a point 82 feet from the base is 0.5 radians. Approximate the height the Diesel Mechanics building to the nearest foot.

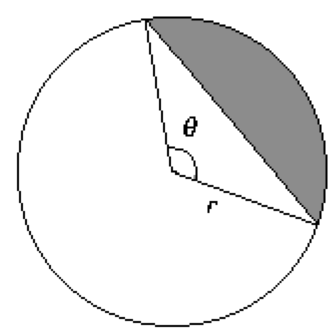
5. Find x correct to four significant digits:



6. Find the area of the triangle at right:

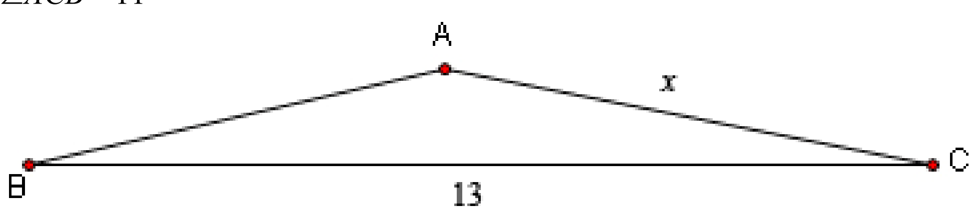


7. If $\theta = \frac{2\pi}{3}$ find the following. Approximate to four significant digits if necessary.
   a. $\tan^2(\theta)$
   b. $\tan(\theta^2)$



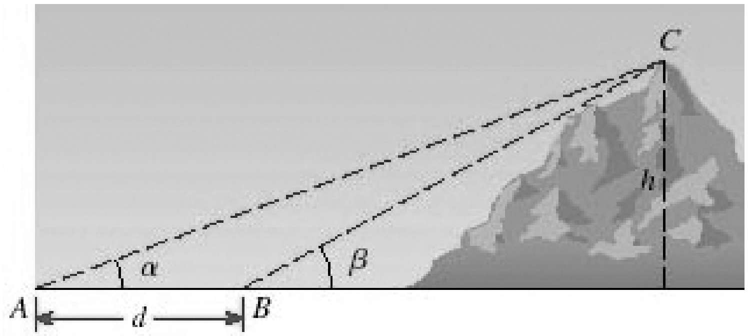
8. Find the area of the shaded region in the figure if $r = 13$ and $\theta = 110°$.

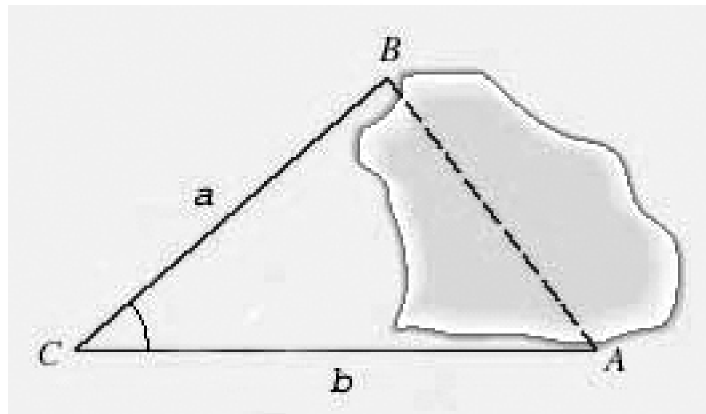
9. Use the law of sines to find the value of x in the diagram below. Assume that $\angle ABC = 13^\circ$ and $\angle ACB = 11^\circ$



10. Sketch the triangle with $\angle A = 32^\circ$, $\angle C = 68^\circ$ and b = 13.11, then solve the triangle.

11. To calculate the height of a mountain, angles $\alpha = 11°$, $\beta = 13°$ and $d = 311$ ft are measured. Use the formula $h = d\dfrac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$ to calculate the height.





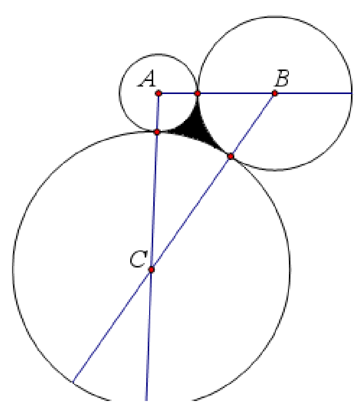
12. To find the distance across a lake, a surveyor has taken the measurements a = 11 mi., b = 13 mi. and $\angle C = 40^\circ$. Find the distance across the lake using this information. Round to 2 significant digits.

13. A toy bicycle with one wheel of diameter 11cm and a bigger wheel with diameter 13cm is rolling along so that the big wheel is rolling at 10 rotations per minute.
 a. What is the angular speed of the little wheel?
 b. What is the linear speed of the bicycle?

14. Suppose we have vectors and $\vec{v} = 11\hat{i} - 13\hat{j}$
 a. Draw and label these vectors together in the x-y plane, assuming each has its initial point at (0,0).
 b. Find the angle between these two vectors.
 c. Find the length of $\vec{u}$ and the length of $\vec{v}$.
 d. Find the lengths of $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$

15. Suppose $\vec{v} = 11\hat{i} - 13\hat{j}$. Find a value of b so that the vector $\vec{u} = 10\hat{i} + b\hat{j}$ is orthogonal to v.

16. Three circles with radii 3, 5 and 12 are externally tangent to one another, as shown in the figure at right.



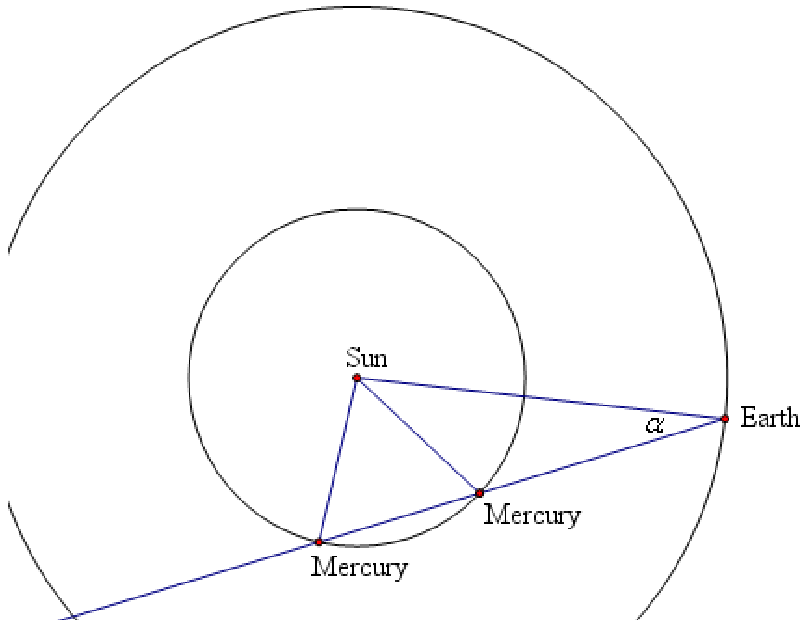
 a. Show that $\angle A = 90^\circ$.

 b. Approximate to the nearest hundredth of a degree measures for $\angle B$ and $\angle C$, interior to triangle ABC.

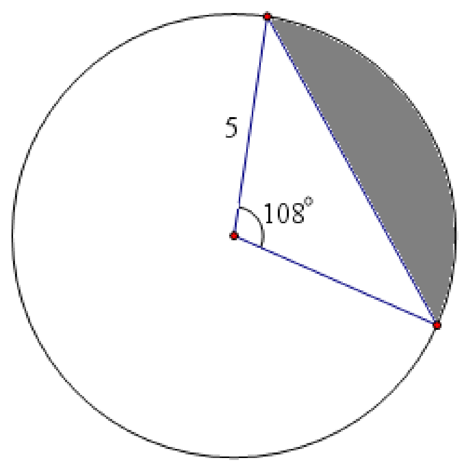
 c. Find the area of the shaded region between the three circles.

17. A tricycle with little wheels of diameter 10cm and a big wheel with diameter 40cm is rolling along so that the big wheel is rolling at 35 rotations per minute.
    a. What is the angular speed of the little wheels?
    b. What is the linear speed of the tricycle? α

18. The elongation α for Mercury is the angle formed by the planet, Earth and Sun, as shown in the diagram at right. Assume the distance from Mercury to the sun is 0.387 AU (38.7% of the distance from Earth to Sun) and that $\alpha = 18^\circ$. Find the possible distances from Earth to Mercury.



19. Find the area of the shaded region in the figure at right.



20. Approximate to the nearest hundredth of a degree, the interior angles of a triangle with sides 4, 5 and 6.

21. Suppose an interior angle of a triangle measures 1 radian and is nested between sides of lengths 11 and 14. What is the length of the side opposite that angle?

22. Suppose we have vectors and $\vec{v} = 2\hat{i} - 3\hat{j}$
    a. Draw and label these vectors together in the *x-y* plane, assuming each has its initial point at (0,0).
    b. Find the angle between these two vectors.
    c. Find the length of $\vec{u}$ and the length of $\vec{v}$.
    d. Find the lengths of $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$