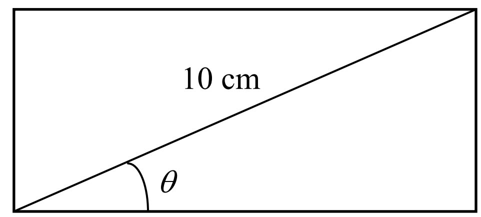
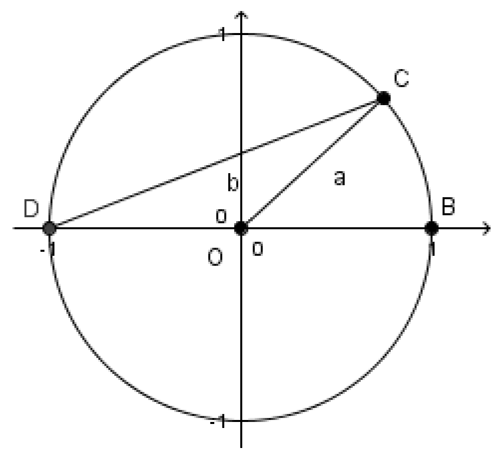
Math 5 – Trigonometry – Final Exam – Spring 2009 NAME________________________________

Show your work for credit. Write all responses on separate paper. There are 13 problems, all weighted equally. Your 3 lowest scoring answers problem will not be counted in your score, so you need only do 10 of the 13.

1. A rectangle's diagonal has a length of 10 cm and an angle of elevation of θ as shown to the right. Find the area of the rectangle as a function of θ.



2. In the figure at right, arc $\overset{\frown}{BC}$ subtends central angle $\angle BOC$ and circular angle $\angle BDC$. Use the facts that ΔCOD is isosceles and that $\angle BOC$ and $\angle COD$ are supplementary to proved that $\angle BOC$ is twice as big as $\angle BDC$. Try to present your response as a proof, supplying justifications for all your claims and logically deducing the desired result.



3. Alexis is looking for a mystery angle β. She knows that $\sin(\beta) = -\frac{1}{2}$. She also knows that β is between 3π radians and $\frac{7\pi}{2}$ radians.
   a. What is β? Give your answer in radian measure.
   b. What is β in degree measure?

4. For each of the following, draw the position of the terminal point corresponding to the input angle on the unit circle and give the (x,y) coordinates of that terminal point. Then form the proper ratio to compute the value of the function, if possible. If not possible, write "does not exist."

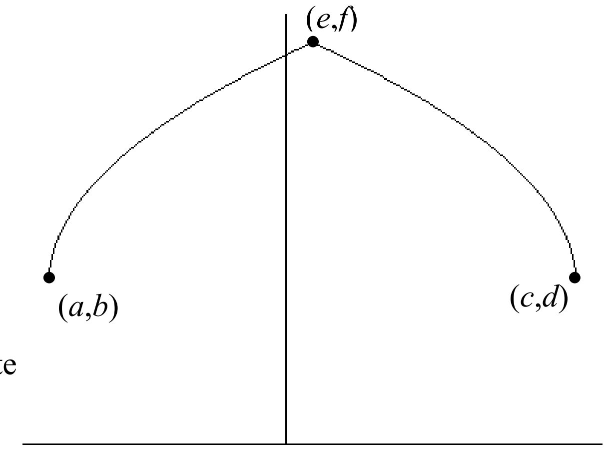
a. $\cos(0)$	c. $\cos\left(\frac{9\pi}{2}\right)$	e. $\sin(390^\circ)$ \
b. $\cos(-60^\circ)$	d. $\sin\left(\frac{15\pi}{4}\right)$	f. $\tan\left(\frac{3\pi}{2}\right)$

5. A particular geometric object drawn in the Cartesian plane satisfies the description "Every point (x,y) on the object is the same distance from the point (2,5) as it is from the x-axis."
   a. Translate this description into an equation in x and y and simplify the equation to show that it is $10y = x^2 - 4x + 29$.
   b. Complete the square on the right side of this equation and then divide both sides by 10 to put this equation in the form $y = a(x-h)^2 + k$
   c. Sketch a reasonably accurate graph of this equation. Label the vertex and at least one pair of points that are symmetric about the line of symmetry.

6. Show a clear argument that the function $f(x)=\dfrac{\sin x+5\tan x}{x^3+x}$ is an even function (its graph is symmetric about the y-axis). Recall that f is even if $f(-x)=f(x)$ for all x in the domain of f.

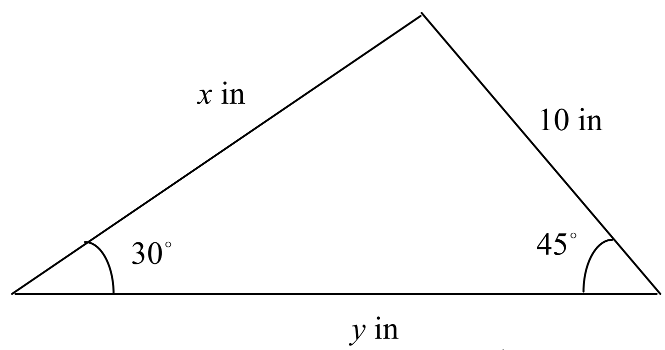
7. Sketch a nice graph of the function $g(x)=10+2\cos\left(\dfrac{\pi}{4}x\right)$. Be sure that the range and the period are evident from looking at the graph.

8. The graph of the function $f(x)=\sqrt{20-|x-2|}+3$ is shown to the right. Find the values of *a, b, c, d, e,* and *f.*

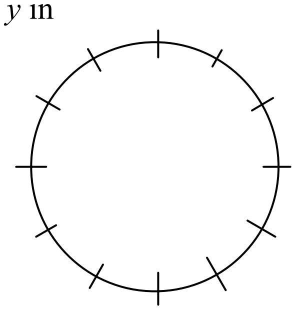


9. Suppose $\cos\theta=\dfrac{5}{13}$ and $270^\circ<\theta<360^\circ$.
   a. Use the unit circle to draw the angle θ reasonably accurate
   b. Find $\sin\theta$
   c. Find $\tan\theta$ and $\sec\theta$

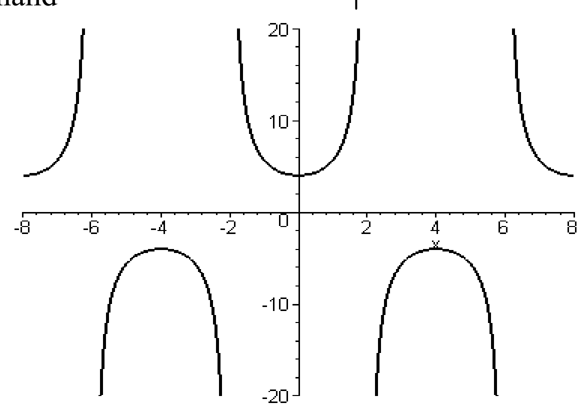
10. a. In the triangle to the right, use the law of sines to find the value of x.
    b. Use the answer to part a) and the law of cosines to find the value of y. Your answer will involve the cosine of an unfamiliar angle, so you may leave your answer in terms of this cosine.



11. The minute hand of a clock does 1 rotation per 60 minutes. This is an angular velocity of $\dfrac{2\pi}{60}\,\dfrac{radians}{\min}=\dfrac{\pi}{30}\;rad/\min$.
    a. If the minute hand is 6 inches long, what is the linear velocity of the tip of the minute hand?
    b. How far (in inches) does the tip of the minute hand travel in 20 minutes?
    c. What is the area of the sector that is swept out by the minute hand in 20 minutes?



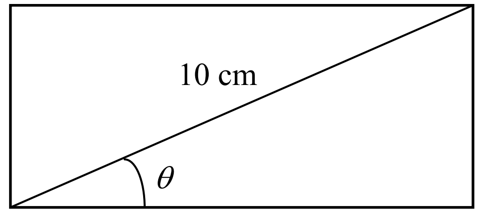
12. To the right is the graph of the function $f(x)=a\sec(bx)$. Use the graph and your understanding of the secant function to determine the values of a and b.



13. Find the standard form of the rectangular equation for the hyperbola parameterized by $x=\sec(t)$ and $y=2\tan(t)$. Find the vertices, foci and asymptotes of the hyperbola and illustrate these in a carefully constructed graph.

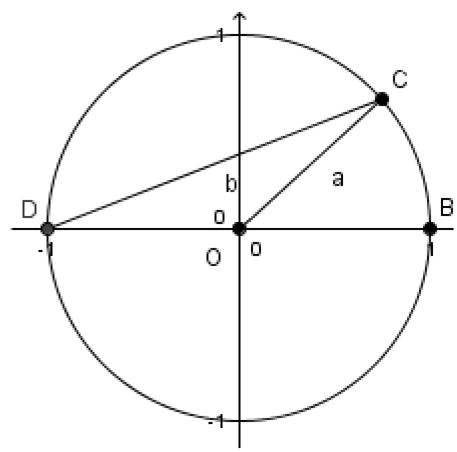
Math 5 – Trigonometry – Final Exam Solutions – Spring 2009

1. A rectangle's diagonal has a length of 10 cm and an angle of elevation of θ as shown to the right. Find the area of the rectangle as a function of θ.



   SOLN: Let h = height and w = width. Then $h = 10\sin(\theta)$ and $w = 10\cos(\theta)$ so the area is $A(\theta) = 100\sin(\theta)\cos(\theta)$.

2. In the figure at right, arc $\widehat{BC}$ subtends central angle $\angle BOC$ and circular angle $\angle BDC$. Use the facts that ΔCOD is isosceles and that $\angle BOC$ and $\angle COD$ are supplementary to prove that $\angle BOC$ is twice as big as $\angle BDC$. Try to present your response as a proof, supplying justifications for all your claims and logically deducing the desired result.



   SOLN

Claim	Reason
1. ΔCOD is isosceles	1. $OD = OC$ are radii of the same circle.
2. $\angle D = \angle C$	2. Base angles of an isosceles triangle are congruent.
3. $\angle BOC + \angle COD = 180^\circ$	3. Supplementary angles
4. $2\angle D + \angle COD = 180^\circ$	4. Sum of interior angles of a triangle and substitution
5. $\angle BOC = 2\angle D$	5. Subtraction, reflective property of = and transitive property of =.
QED	

3. Alexis is looking for a mystery angle β. She knows that $\sin(\beta) = -\frac{1}{2}$.

   She also knows that β is between 3π radians and $\frac{7\pi}{2}$ radians.

   a. What is β? Give your answer in radian measure.

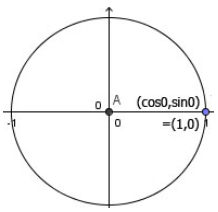
   SOLN: So β is in QIII and the reference angle is $\pi/6$. Thus $\beta = 3\pi + \frac{\pi}{6} = \frac{19\pi}{6}$

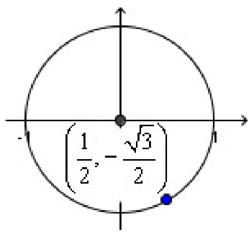
   b. What is β in degree measure?

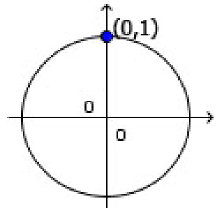
   SOLN: $\beta = \frac{19\pi}{6} \cdot \frac{180^\circ}{\pi} = 570^\circ$

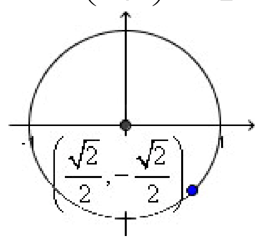
4. For each of the following, draw the position of the terminal point corresponding to the input angle on the unit circle and give the (x,y) coordinates of that terminal point. Then form the proper ratio to compute the value of the function, if possible. If not possible, write “does not exist.”

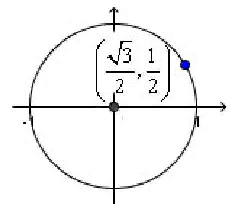
a. $\cos(0) = 1$ b. $\cos\left(-60^\circ\right) = \frac{1}{2}$ c. $\cos\left(\frac{9\pi}{2}\right) = 0$ d. $\sin\left(\frac{15\pi}{4}\right) = \frac{-\sqrt{2}}{2}$ e. $\sin\left(390^\circ\right) = \frac{1}{2}$ f. $\tan\left(\frac{3\pi}{2}\right)$





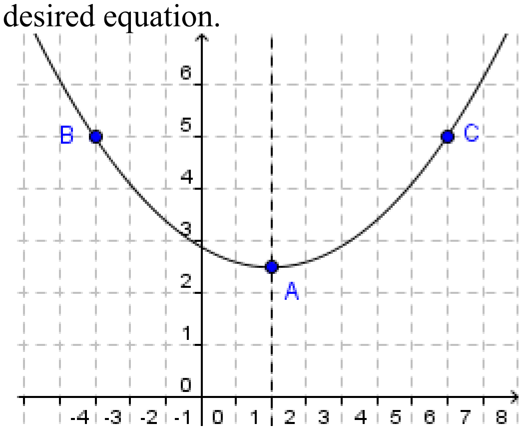






The point is the same as in (b) but the ratio does not exist.

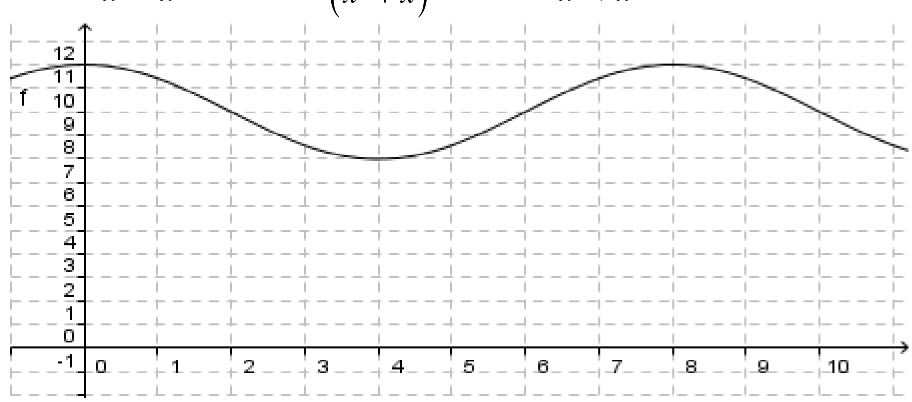
5. A particular geometric object drawn in the Cartesian plane satisfies the description “Every point (x,y) on the object is the same distance from the point (2,5) as it is from the x-axis.”
   a. Translate this description into an equation in x and y and simplify the equation to show that it is $10y = x^2 - 4x + 29$.
   SOLN: Equating the squares of the distances leads to $y^2 = (x-2)^2 + (y-5)^2$. Expanding, collecting like terms and adding $10y$ to both sides then leads quickly to the desired equation.
   b. Complete the square on the right side of this equation and then divide both sides by 10 to put this equation in the form $y = a(x-h)^2 + k$
   SOLN:
   $10y = x^2 - 4x + 29 = (x-2)^2 + 25 \Leftrightarrow y = \frac{1}{10}(x-2)^2 + \frac{5}{2}$
   c. Sketch a reasonably accurate graph of this equation. Label the vertex and at least one pair of points that are symmetric about the line of symmetry.
   SOLN: Vertex A(2,2.5) and symmetric points B(–3,5) and C(–3,5)



6. Show a clear argument that the function $f(x) = \frac{\sin x + 5\tan x}{x^3 + x}$ is an even function (its graph is symmetric about the y-axis). Recall that f is even if $f(-x) = f(x)$ for all x in the domain of f.

SOLN: $f(-x) = \frac{\sin(-x) + 5\tan(-x)}{(-x)^3 + (-x)} = \frac{-\sin x - 5\tan x}{-x^3 - x} = \frac{-(\sin x + 5\tan x)}{-(x^3 + x)} = \frac{\sin x + 5\tan x}{x^3 + x} = f(x)$

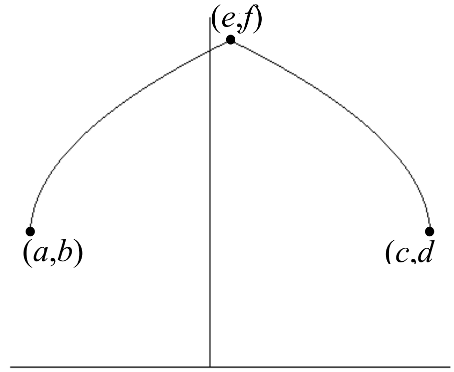
7. Sketch a nice graph of the function $g(x) = 10 + 2\cos\left(\frac{\pi}{4}x\right)$. Be sure that the range and the period are evident from looking at the graph.
SOLN: The amplitude of 2 oscillates from 8 to 12 about a line $y = 10$ with period = 8.



8. The graph of the function $f(x) = \sqrt{20 - |x-2|} + 3$ is shown to the right. Find the values of *a, b, c, d, e,* and *f.*

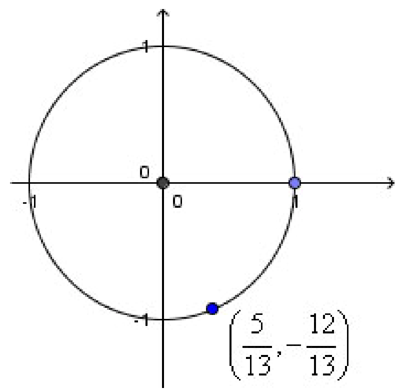
SOLN: The domain is [–18,22] and the range is $\left[3, 3+2\sqrt{5}\right]$

The graph has symmetry about the line $x = 2$. Thus $a = -18$, $b = d = 3$, $c = 22$, $e = 2$ and $f = 3+2\sqrt{5}$.



9. Suppose $\cos\theta = \dfrac{5}{13}$ and $270^\circ < \theta < 360^\circ$.

a. Use the unit circle to draw the angle θ reasonably accurately.
Soln: It's in QIV, as shown to right.



b. Find $\sin\theta$ $\left(\dfrac{5}{13}, -\dfrac{12}{13}\right)$

Soln: The y-coordinate is

$$-\sqrt{1-\left(\frac{5}{13}\right)^2} = -\sqrt{1-\frac{25}{169}} = -\sqrt{\frac{144}{169}} = -\frac{12}{13}$$

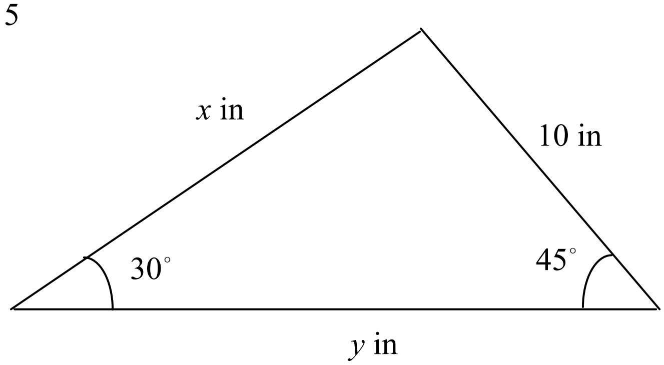
c. Find $\tan\theta$ and $\sec\theta$

SOLN: $\tan\theta = \dfrac{-12/13}{5/13} = -\dfrac{12}{5}$ and $\sec\theta = \dfrac{1}{5/13} = \dfrac{13}{5}$

10. a. In the triangle to the right, use the law of sines to find the value of x.

SOLN:

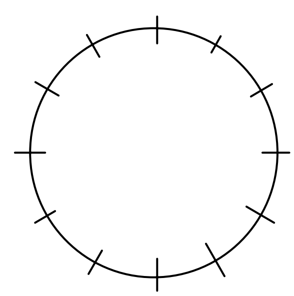
$$\frac{x}{\sin 45^\circ} = \frac{10}{\sin 30^\circ} \Leftrightarrow \frac{x}{1/\sqrt{2}} = \frac{10}{1/2} \Leftrightarrow x = \frac{20}{\sqrt{2}} = 10\sqrt{2}$$



b. Use the answer to part a) and the law of cosines to find the value of y.

SOLN: $y = \sqrt{\left(10\sqrt{2}\right)^2 + 10^2 - 2\left(10\sqrt{2}\right)(10)\cos\left(105^\circ\right)} = \sqrt{300 - 200\sqrt{2}\cos\left(105^\circ\right)}$

11. The minute hand of a clock does 1 rotation per 60 minutes. This is an angular velocity of $\dfrac{2\pi}{60}\dfrac{radians}{\text{min}} = \dfrac{\pi}{30}\ rad/\text{min}$.

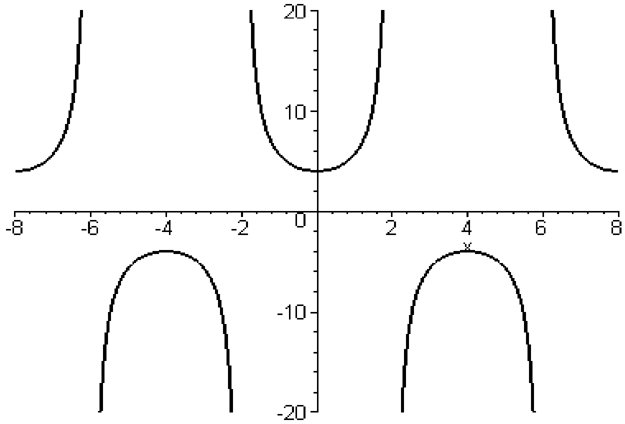


a. If the minute hand is 6 inches long, what is the linear velocity of the tip of the minute hand?
SOLN: $v = \omega r = \pi/5$ cm/min

b. How far (in inches) does the tip of the minute hand travel in 20 minutes?
SOLN: $d = vt = (\pi/5)20 = 4\pi$ cm.

c. What is the area of the sector that is swept out by the minute hand in 20 minutes?
SOLN: $A = r^2\theta/2 = 36(\pi/3) = 12\pi$ cm^2.

12. To the right is the graph of the function $f(x) = a\sec(bx)$. Use the graph and your understanding of the secant function to determine the values of a and b.|



SOLN: $y = 4\sec(\pi x/4)$ so $a = 4$ and $b = \pi/4$

This is deduced by observing that the usual gap in y-values: $(-1,1)$ is stretched by 4 to $(-4,4)$ and the asymptotes at $x = \pi/2$ and $x = -\pi/2$ are stretched to $x = -2$ and $x = 2$.

13. Find the standard form of the rectangular equation for the hyperbola parameterized by $x = \sec(t)$ and $y = 2\tan(t)$. Find the vertices, foci and asymptotes of the hyperbola and illustrate these in a carefully constructed graph.
SOLN:
$\sec^2(t) - \tan^2(t) = 1$ leads to $x^2 - \frac{y^2}{4} = 1$ so the vertices are at (1,0) and (–1,0) the foci are at $\left(\pm\sqrt{5},0\right)$, the asymptotes are $y = 2x$ and $y = -2x$.