Math 5 – Trigonometry – fall ’10 – Chapter 5 Test – Fair Game Problems

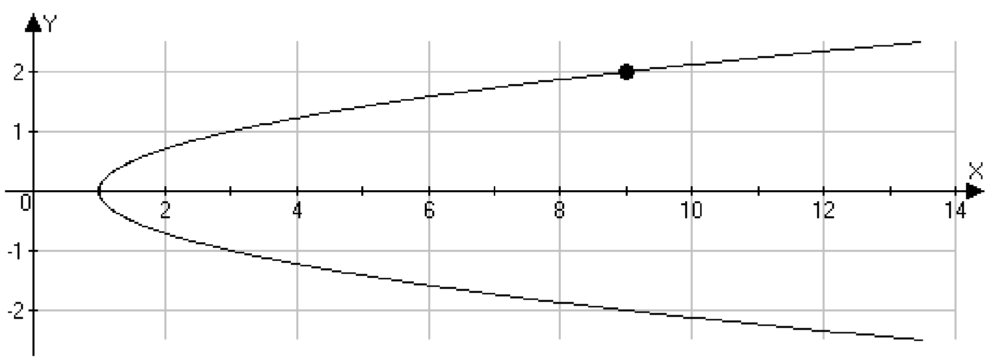
1. Find an equation for the parabola with vertex at (0,0) and directrix $x = 2$. What is the focal diameter? Construct a careful graph.

2. Consider the conic described by $4x^2 + y^2 = 4$.
   a. Where are the vertices? Give coordinates.
   b. Where are the foci? Give coordinates.
   c. Find the eccentricity.
   d. Write parameteric equations for this conic.
   e. Construct a careful graph showing the key features.

3. Consider the conic described by $4x^2 - y^2 = -4$.
   a. Where are the vertices? Give coordinates.
   b. Where are the foci? Give coordinates.
   c. Write equations for the asymptotes.
   d. Write parameteric equations for this conic.
   e. Construct a careful graph showing the key features.

In 4 – 6, write an equation for the conic whose graph is shown.

4.



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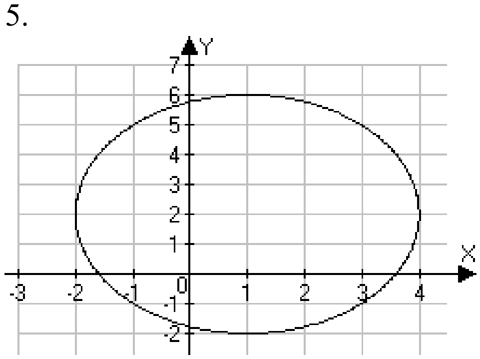

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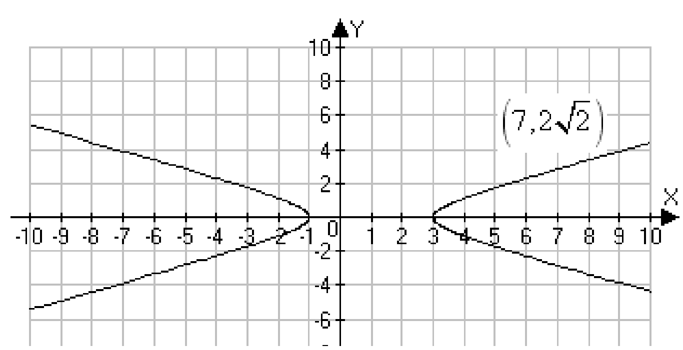


7. Write in standard form and sketch a graph: $x^2 + 2y^2 + 3x + 4y + 5 = 0$

8. Write in standard form and sketch a graph: $x^2 - 2y^2 + 3x - 4y + 5 = 0$

9. Write an equation for the hyperbola with foci at $\left(1 \pm \sqrt{2}, 0\right)$ and asymptotes $y = 1 \pm x$

10. Write a polar equation of a conic with the focus at the origin and the given data.

a. Parabola, vertex at $(r, \theta) = (1, \pi/2)$.

b. Ellispse, eccentricity 0.05, vertex at $(r, \theta) = (2, \pi)$.

c. Hyperbola, eccentricity 20 and directrix $r = -4\csc\theta$.

11. Find the eccentricity and identify the conic with polar equation $r = \dfrac{5}{2 - 3\sin\theta}$.

Find the vertices, foci and asymptotes and sketch a graph showing these.

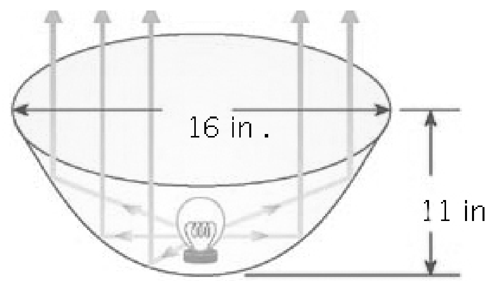
12. Consider the parametric curve given by

$$x = 1 + 2\cos\theta$$
$$y = 4 + 3\sin\theta$$
$$0 \le \theta \le 2\pi$$

a. Construct a careful graph for the curve.

b. Eliminate the parameter θ to obtain an equation for this curve in rectangular coordinates.

13. A light bulb is to be placed at the focus of a parabolic dish as shown in the figure at right. How high above the bottom should the light be placed?



14. Find an equation for the ellipse with foci $(\pm 10, 0)$ and vertices $(\pm 11, 0)$.

15. Find the vertices, foci, and asymptotes of the hyperbola $144y^2 - 36x^2 = 3600$ and sketch a graph illustrating these features.

16. Find an equation for the hyperbola with asymptotes $y = \pm\frac{2}{3}x$ and vertices at

a. $(0, \pm 3)$ b. $(\pm 3, 0)$

17. Complete the square to determine whether the equation represents an ellipse, a parabola, or a hyperbola. If the graph is an ellipse, find the center, foci, eccentricity and endpoints of the major and minor axes. If it is a parabola, find the vertex, focus and directrix. If it is a hyperbola, find the center, foci, vertices and asymptotes.

a. $16x^2 - 96x + 9y^2 = 0$

b. $36y^2 - x^2 - 8x - 52 = 0$

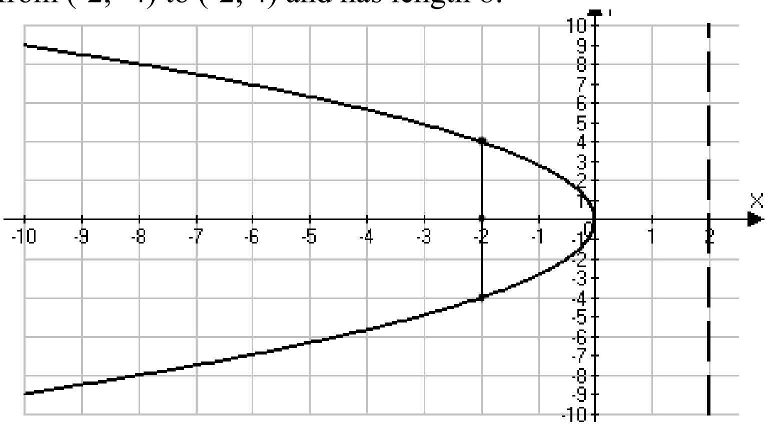
c. $x^2 - 8x - 32y - 240 = 0$

18. Find parametric equations to describe the hyperbola $4(x-1)^2 - 25(y-4)^2 = 100$

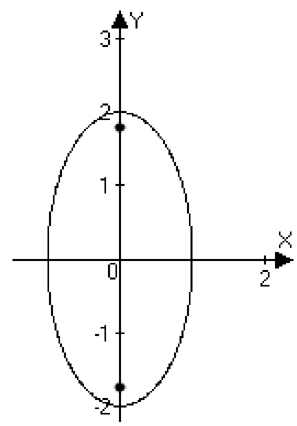
19. Write the equation for the conic section described by $\begin{matrix} y = 1 + 8\sin(2t) \\ x = 9 + 72\cos(2t) \end{matrix}$ in rectangular form.

Math 5 – Trigonometry – fall '10 – Chapter 5 –Fair Game Problems Solutions

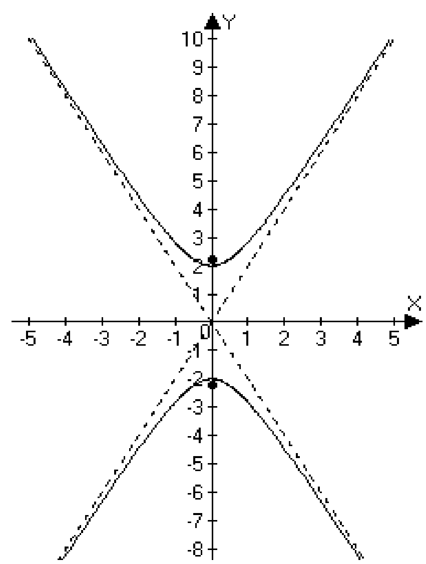
1. Find an equation for the parabola with vertex at (0,0) and directrix $x = 2$. What is the focal diameter? Construct a careful graph.
   SOLN: with $p = -2$ we have the equation $4px = -8x = y^2$ and the focal diameter stretches from (-2, -4) to (-2, 4) and has length 8:



2. Consider the conic described by $4x^2 + y^2 = 4$.
   a. $4x^2 + y^2 = 4 \Leftrightarrow x^2 + \frac{y^2}{4} = 1$ has vertices at $(\pm1,0),(0,\pm2)$.
   b. $c^2 = a^2 - b^2 = 4 - 1 = 3 \Rightarrow c = \sqrt{3}$ so the foci are at $(0,\pm\sqrt{3})$
   c. The eccentricity is $\frac{c}{a} = \frac{\sqrt{3}}{2} \approx 87\%$
   d. Parametric equations: $x = \cos t \qquad y = 2\sin t$
   e. A careful graph showing the key features is shown at right:

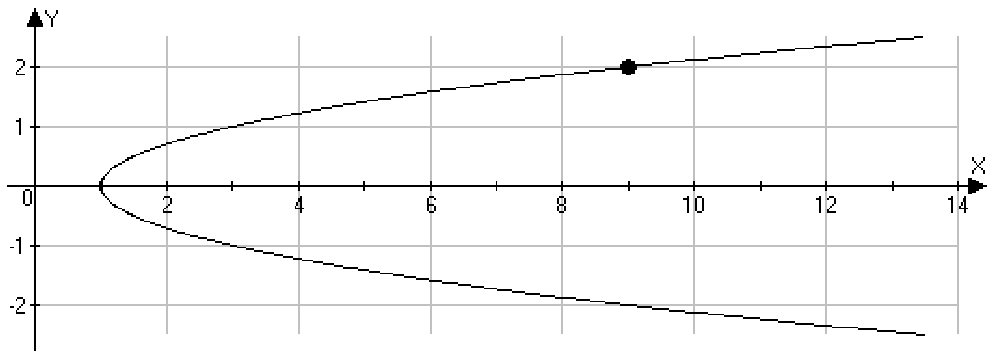


3. Consider the conic described by $4x^2 - y^2 = -4$.
   a. $4x^2 - y^2 = -4 \Leftrightarrow \frac{y^2}{4} - x^2 = 1$ has vertices at $(0,\pm2)$
   b. Here $c^2 = a^2 + b^2 = 4 + 1 = 5 \Rightarrow c = \sqrt{5}$ so the foci are at $(0,\pm\sqrt{5})$
   c. The asymptotes are along $y = \pm2x$
   d. Parametric equations: $x = \tan t \qquad y = 2\sec t$
   e. A careful graph showing the key features is shown at right.

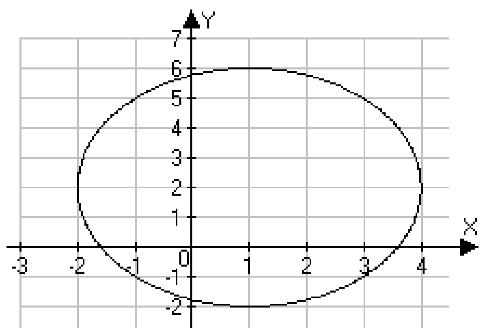


In 4 – 6, write an equation for the conic whose graph is shown.

4. In the parabola below, the vertex is at (1,0) so the equation is $4p(x-1)=y^2$. Since the parabola passes through (9,2) we must have $4p(9-1)=2^2 \Leftrightarrow p=\frac{1}{8}$, so the parabola's equation can be written $\frac{1}{2}(x-1)=y^2$

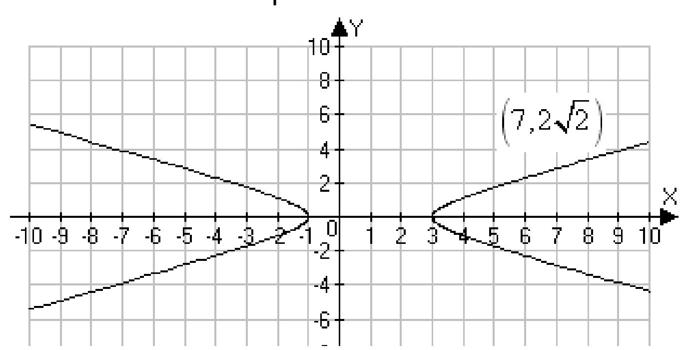


5. The ellipse is centered at (1,2) with $a = 4$, $b = 3$: $\frac{(x-1)^2}{9}+\frac{(y-2)^2}{16}=1$



6. The hyperbola is centered at (1,0) , has vertices at a distance $a = 2$ from the center. Thus we can write $\frac{(x-1)^2}{4}-\frac{y^2}{b^2}=1$. Now plug in the coordinates of the given point: $\frac{(7-1)^2}{4}-\frac{\left(2\sqrt{2}\right)^2}{b^2}=1 \Leftrightarrow 9-\frac{8}{b^2}=1$

Thus $b = 1$ and $\frac{(x-1)^2}{4}-y^2=1$



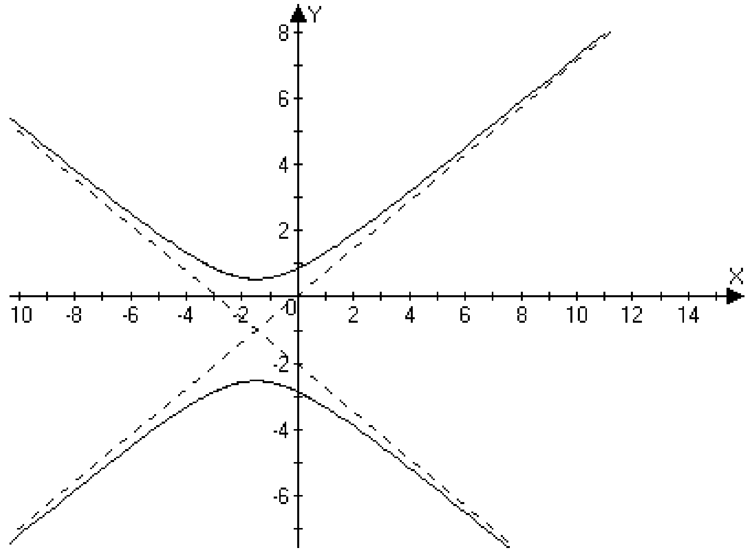
7. Write in standard form and sketch a graph: $x^2+2y^2+3x+4y+5=0$

SOLN: $x^2+3x+\frac{9}{4}+2\left(y^2+2y+1\right)=-5+\frac{9}{4}+2 \Leftrightarrow \frac{-\left(x+\frac{3}{2}\right)^2}{3/4}-\frac{(y+1)^2}{3/8}=1$. Since the left side of the equation is less than or equal to zero for all x and y, there is no graph in the real plane.

8. Write in standard form and sketch a graph: $x^2-2y^2+3x-4y+5=0$

SOLN: $x^2+3x+\frac{9}{4}-2\left(y^2+2y+1\right)=-5+\frac{9}{4}-2 \Leftrightarrow \frac{(y+1)^2}{19/8}-\frac{\left(x+\frac{3}{2}\right)^2}{19/4}=1$.

This means the vertices of the hyperbola are at $\left(-\frac{3}{2},-1\pm\frac{\sqrt{38}}{4}\right)$ and the asymptotes are along $y+1=\pm\frac{\sqrt{2}}{2}\left(x+\frac{3}{2}\right)$



9. Write an equation for the hyperbola with foci at $\left(1\pm\sqrt{2},0\right)$ and asymptotes $y=1\pm x$

SOLN: The center is at $(1,0)$ and $c^2 = 2$. From the slopes of the asymptotes we know $a = b$ so $c^2=a^2+b^2 \Leftrightarrow 2=2a^2$ which means that $a = b = 1$. Thus the equation is $(x-1)^2-y^2=1$

10. Write a polar equation of a conic with the focus at the origin and the given data.

a. Parabola, vertex at $(r, \theta) = (1, \pi/2)$. SOLN: The rectangular coordinates for the vertex are (0,1) and since the focus is at (0,0), $p = 1$, and the directrix must be $y = 2$. The rectangular equation is thus $-4(y-1)=x^2$ $r=\frac{2}{1+\sin\theta}$ or $r=\frac{-2}{1-\cos\left(\theta-\frac{\pi}{2}\right)}$, either of these will be a good polar form for the parabola. We can easily convert to rectangular coordinates:

$$=\frac{-2}{1-\sin\theta}\Rightarrow r-r\sin\theta=-2\Leftrightarrow r=y-2\Rightarrow x^2+y^2=y^2-4y+4\Leftrightarrow \boxed{-4(y-1)=x^2}$$

b. Ellispse, eccentricity 0.05, vertex at $(r, \theta) = (2, \pi)$.
SOLN: Here the rectangular coordinates of one vertex are (–2,0).
We can plug the polar coordinates, $(r, -\theta) = (2, \pi)$ values into the polar form and get $2=\frac{0.05d}{1-0.05\cos\pi}=\frac{d}{21}\Rightarrow d=42$, so the polar form of the equation is $r=\frac{2.1}{1-0.05\cos\theta}$. At this point, the question has been answered, but there are some obvious follow-up questions, such as "What is the rectangular equation form for this ellipse?" We can find the center as the average of the two x-intercepts: $\frac{1}{2}\left(\frac{2.1}{1-0.05}-2\right)=\frac{1}{2}\left(\frac{210}{95}-2\right)=\frac{2}{19}$ so that $c=\frac{2}{19}$ and $a=\frac{40}{19}$, whence $b^2=a^2-c^2=\left(\frac{2\sqrt{399}}{19}\right)^2$ and the rectangular form of the equation is

$$\left(\frac{19}{40}\right)^2\left(x-\frac{2}{19}\right)^2+\left(\frac{19}{2\sqrt{399}}\right)^2y^2=1$$

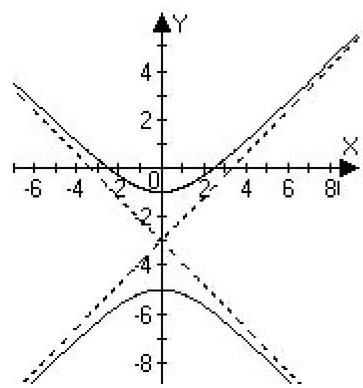
c. Hyperbola, eccentricity 20 and directrix $r = -4\csc\theta$.
SOLN: The rectangular form for the equation of the directrix is $y = -4$.
A polar form of the equation is $r = \dfrac{80}{1-20\sin\theta}$. This means the vertices are at
$r\left(\dfrac{\pi}{2}\right) = \dfrac{80}{-19} = \dfrac{-80}{19}$ and $r\left(-\dfrac{\pi}{2}\right) = \dfrac{80}{21}$ which correspond to the y-intercepts, $\left(0, -\dfrac{80}{19}\right)$ and
$\left(0, -\dfrac{80}{21}\right)$. Thus the center of the hyperbola the average of these:
$-k = -c = \dfrac{1}{2}\left(-\dfrac{80}{19} - \dfrac{80}{21}\right) = -\dfrac{1}{2}\left(\dfrac{80(21+19)}{19*21}\right) = \dfrac{-1600}{399}$, which means the distance from the
center to vertex is $a = \dfrac{80}{19} - \dfrac{1600}{399} = \dfrac{80}{399}$ from which we get $b^2 = c^2 - a^2 =$
$\dfrac{1600^2 - 80^2}{399^2} = \dfrac{2553600}{399^2} = \left(\dfrac{80}{\sqrt{399}}\right)^2$ whence the equation

$$\left(\frac{399}{80}\right)^2\left(y + \frac{1600}{399}\right)^2 + \left(\frac{\sqrt{399}}{80}\right)^2 y^2 = 1.$$

11. Find the eccentricity and identify the conic with polar equation $r = \dfrac{5}{2-3\sin\theta}$. Find the vertices and foci and sketch a graph.

r	$\frac{5}{2}$	-5	$\frac{5}{2}$	1
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

SOLN: $r = \dfrac{5}{2-3\sin\theta} = \dfrac{2.5}{1-1.5\sin\theta}$ shows $e = 1.5$, so the conic is an hyperbola. Tabulate the extreme points in polar coordinates to determine the vertices are at (0,–5) and (0,–1). The center is (0,–3), so $a = 2$ whence $c = ae = 3$ and the foci are at (0, 0) and (0, –6) and since $b = \sqrt{c^2 - a^2} = \sqrt{9-4} = \sqrt{5}$, the asymptotes are along $y + 3 = \pm\dfrac{2\sqrt{5}}{5}x$

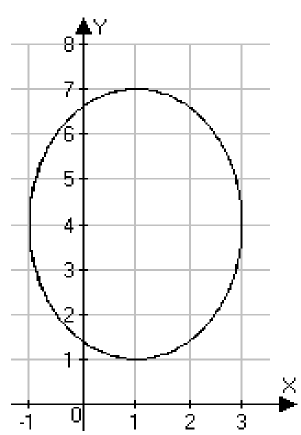


12. Consider the parametric curve given by

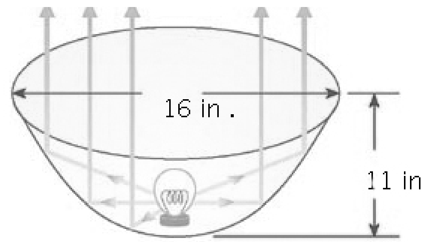
$$x = 1 + 2\cos\theta$$
$$y = 4 + 3\sin\theta$$
$$0 \le \theta \le 2\pi$$

a. Construct a careful graph for the curve.
b. Eliminate the parameter θ to obtain an equation for this curve in rectangular coordinates.

$$\frac{(x-1)^2}{4} + \frac{(y-4)^2}{9} = 1$$



13. A light bulb is to be placed at the focus of a parabolic dish as shown in the figure at right. How high above the bottom should the light be placed?
SOLN: If the parabola is opening upwards from a vertex at (0,0), then it has the form $4py = x^2$, whence $44p = 64$ and the distance from the focus to the vertex is $p = 16/11$.

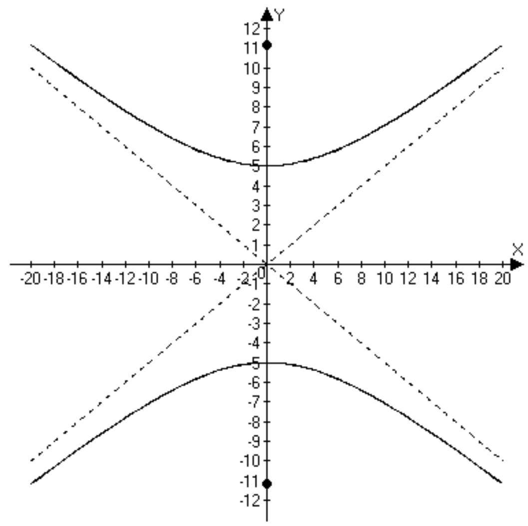


14. Find an equation for the ellipse with foci $(\pm 10, 0)$ and vertices $(\pm 11, 0)$.

SOLN: $b^2 = a^2 - c^2 = 11^2 - 10^2 = 21$ so the equation is $\frac{x^2}{121} + \frac{y^2}{21} = 1$

15. Find the vertices, foci, and asymptotes of the hyperbola $144y^2 - 36x^2 = 3600$ and sketch a graph illustrating these features.

SOLN: $144y^2 - 36x^2 = 3600 \Leftrightarrow \frac{y^2}{25} - \frac{x^2}{100} = 1$ has vertices at foci at $(0, \pm 5)$ $(0, \pm 5\sqrt{5})$ The asymptotes are $y = \pm \frac{1}{2}x$



16. Find an equation for the hyperbola with asymptotes $y = \pm \frac{2}{3}x$ and vertices at

a. $(0, \pm 3)$ So we know the ratio of $b/a = 2/3$ and that $b = 3$. Thus $a = 9/2$ and the equation is $\frac{y^2}{9} - \frac{4x^2}{81} = 1$

b. $(\pm 3, 0)$ Here $a = 3$ so $b = 2$ and the equation is simply $\frac{x^2}{9} - \frac{x^2}{4} = 1$

17. Complete the square to determine whether the equation represents an ellipse, a parabola, or a hyperbola. If the graph is an ellipse, find the center, foci, eccentricity and endpoints of the major and minor axes. If it is a parabola, find the vertex, focus and directrix. If it is a hyperbola, find the center, foci, vertices and asymptotes.

a. $16x^2 - 96x + 9y^2 = 0 \Leftrightarrow 16(x^2 - 6x + 9) + 9y^2 = 144 \Leftrightarrow \frac{(x-3)^2}{9} + \frac{y^2}{16} = 1$ is an ellipse with center (3,0), endpoints of minor axes at (0,0) and (6,0) and major axes at (3, –4) and (3, 4) with foci at $(3, \pm\sqrt{7})$ and eccentricity $\sqrt{7}/4$

b. $36y^2 - x^2 - 8x - 52 = 0 \Leftrightarrow 36y^2 - (x-4)^2 = 36 \Leftrightarrow y^2 - \frac{(x-4)^2}{36} = 1$ is an ellipse with center (-4,0), vetices (-4,–1) and (-4,1), foci at $(-4, \pm\sqrt{37})$ and asymptotes $y = \pm\frac{1}{6}(x-4)$.

c. $x^2 - 8x - 32y - 240 = 0 \Leftrightarrow 32(y+8) = (x-4)^2$ is a parabola with vertex (4, –8), focus at (4,0), directrix along $y = -16$.

18. Find parametric equations to describe the hyperbola $4(x-1)^2 - 25(y-4)^2 = 100$

SOLN: $x = 1 + 5\sec t$ and y= $4 + 2\tan t$

19. Write the equation for the conic section described by $\begin{matrix} y = 1 + 8\sin(2t) \\ x = 9 + 72\cos(2t) \end{matrix}$ in rectangular form.

$$\frac{(x-9)^2}{5184} + \frac{(y-1)^2}{64} = 1$$