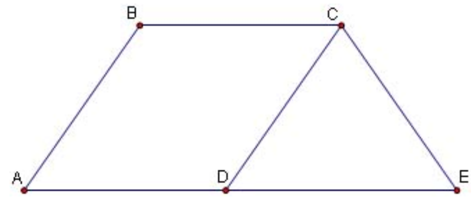
Math 5 – Final Exam – Fall 2010 Name____||

Instructions: Write all responses on these pages or attach additional pages, as needed.

1. Give a two column proof.

   Given: *ABCD* is a parallelogram, $\angle CDE \cong \angle CED$ and $AD \cong CE$.

   Prove: *ABCD* is a rhombus.



2. Consider the parabola described by the function $f(x) = 6 + 6x - x^2$.
   a. Find the vertex of the parabola and write the function in the vertex form: $y = a(x - h)^2 + k$
   b. Find the x-intercepts of the parabola.
   c. Find where the parabola intersects the line $y = -2x + 9$
   d. Sketch a graph of the parabola $f(x) = 6 + 6x - x^2$ and the line $y = -2x + 9$ showing where they intersect.

3. If $\tan t = \frac{3}{4}$ and t is in QIII, then find the following: (HINT: use the unit circle.)

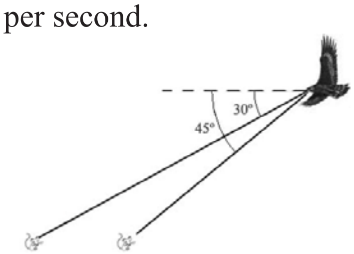
   a. $\csc(t) + \sec(t)$ b. $\sin\left(\frac{\pi}{2} - t\right) + \cos\left(\frac{\pi}{2} - t\right)$

4. A point P moving in simple harmonic motion completes 8 cycles every second. If the amplitude of the motion is 50 cm, find an equation that describes the motion of P as a function of time. Assume the point P is at equilibrium when $t = 0$.

5. Consider the simple harmonic oscillator whose function is $x(t) = -\cos\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$.
   a. Find the amplitude, period and phase shift of oscillation.
   b. Construct a graph showing the coordinates of at least 5 key points of the function.

6. A potter's wheel with radius 4 cm spins at 140 rotations per minute. Find
   a. The angular speed of the wheel in radians per second.
   b. The angular speed of the wheel in degrees per minute.
   c. The linear speed of a point on the rim of the wheel in meters per second.

7. A bird observes the angles of depression to two rodents on the ground to be 45º and 30º, as shown. If the bird is flying at an elevation of 100 meters, find the distance between the rodents.
   a. Consider the triangle with sides of lengths 5, 6 and 8 meters. Solve the triangle. That is, find the degree measures of the interior angles.
   b. Find the area of the triangle.



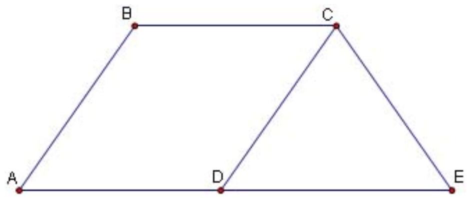
8. Consider the ellipse described by $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the hyperbola described by $x^2 - y^2 = 1$
   a. Sketch a graph showing the ellipse and the hyperbola together.
   b. What are the coordinates of the points where the two conic sections intersect?

9. Consider the ellipse parameterized by $x(t) = 1+2\sin(t)$ and $y(t) = 2 + 3\cos(t)$
   a. Write the standard form for the equation of this ellipse
   b. Complete the table and use the values to sketch a graph for the part of ellipse where $0 \le t \le \pi$

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x					
y					

Math 5 – Final Exam Solutions – Fall 2010

1. Give a two column proof.



Given: *ABCD* is a parallelogram,
$\angle CDE \cong \angle CED$ and $AD \cong CE$.

Prove: *ABCD* is a rhombus.

Proof:

Statement	Reason
$\angle CDE \cong \angle CED$ and $AD \cong CE$.	Given
ΔCDE is isosceles	A triangle is isosceles iff it has two congruent angles
$AD \cong CE$	Definition of isosceles triangle
$AD \cong DC$	Transitivity of congruence
ABCD is a parallelogram	Given
ABCD is a rhombus	Definition of rhombus

2. Consider the parabola described by the function $f(x) = 6 + 6x - x^2$.

a. Find the vertex of the parabola and write the function in the vertex form: $y = a(x-h)^2 + k$

SOLN: $f(x) = -x^2 + 6x + 6 = -(x-3)^2 + 15$ so the vertex is at (3,15).

b. Find the x-intercepts of the parabola.

SOLN: It's easiest to solve the vertex form equation, from which it's readily clear that

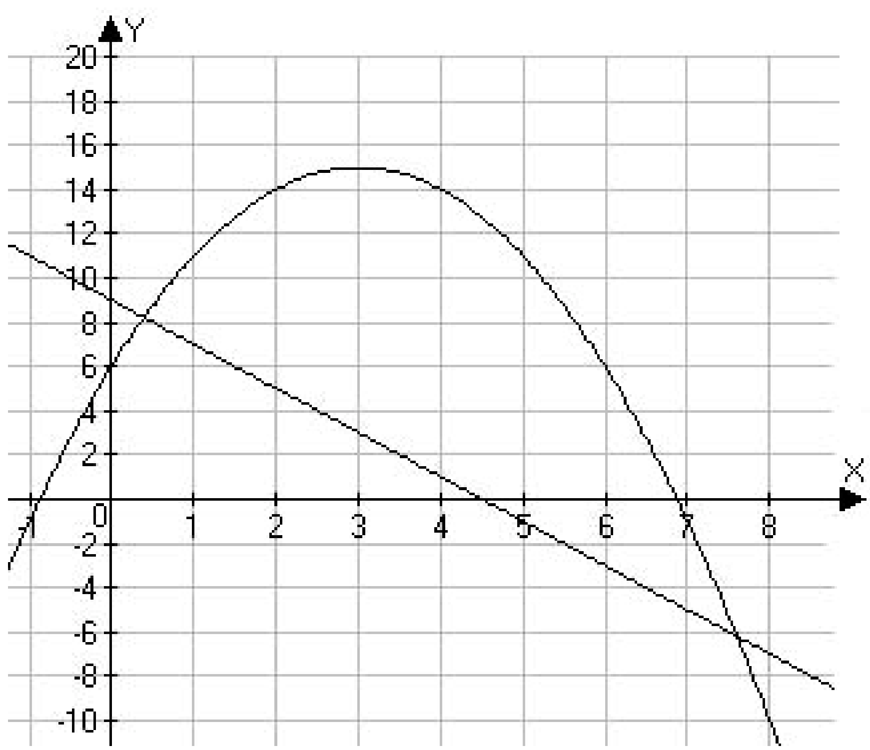
$(x-3)^2 = 15 \Leftrightarrow x = 3 \pm \sqrt{15}$

c. Find where the parabola intersects the line $y = -2x + 9$

SOLN: $-x^2 + 6x + 6 = -2x + 9 \Leftrightarrow x^2 - 8x = -3 \Leftrightarrow (x-4)^2 = 13 \Leftrightarrow x = 4 \pm \sqrt{13}$ so the parabola and the line intersect at $\left(4 \pm \sqrt{13}, 1 \mp 2\sqrt{13}\right) \approx (4 \pm 3.6, 1 \mp 7.2) = (0.4, 8.2)$ or $(7.6, -6.2)$

d. Sketch a graph of the parabola $f(x) = 6 + 6x - x^2$ and the line $y = -2x + 9$ showing where they intersect.

SOLN:



3. If $\tan t = \dfrac{3}{4}$ and t is in QIII, then find the following: (HINT: use the unit circle.)

   a. $\csc(t) + \sec(t)$

      SOLN: This is the 4-3-5 triple in QIII, so sin(t) = –3/5 and cos(t) = –4/5 whence

      $$\csc(t) + \sec(t) = -\frac{5}{3} - \frac{5}{4} = -\frac{35}{12}$$

   b. $\sin\left(\dfrac{\pi}{2} - t\right) + \cos\left(\dfrac{\pi}{2} - t\right)$

      SOLN: $\sin\left(\dfrac{\pi}{2} - t\right) + \cos\left(\dfrac{\pi}{2} - t\right) = \cos(t) + \sin(t) = -\dfrac{3}{5} - \dfrac{4}{5} = -\dfrac{7}{5}$

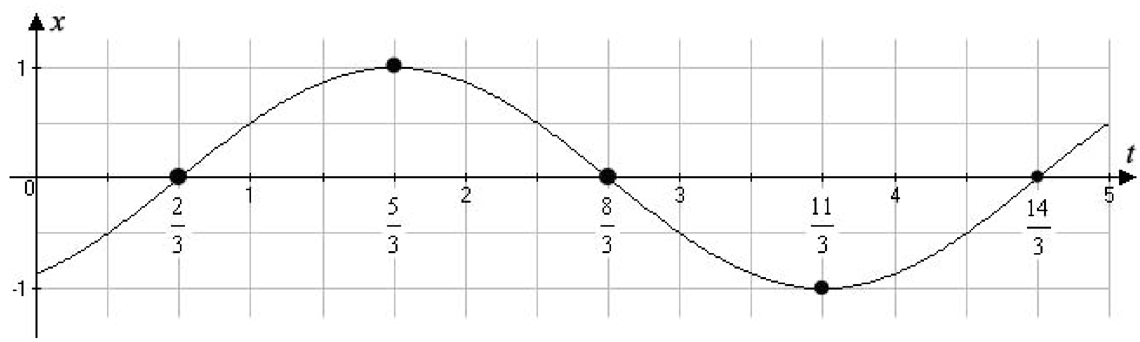
4. A point P moving in simple harmonic motion completes 8 cycles every second. If the amplitude of the motion is 50 cm, find an equation that describes the motion of P as a function of time. Assume the point P is at equilibrium when $t = 0$.
   SOLN: A good simple harmonic oscillator model for this set up is $P(t) = A\sin(\omega t)$.
   8 Hz means $\omega = 2\pi * 8$ and $A = 50$ so $P(t) = 50\sin(16\pi t)$ does the job.

5. Consider the simple harmonic oscillator whose function is $x(t) = -\cos\left(\dfrac{\pi}{2}t + \dfrac{\pi}{6}\right)$.

   a. Find the amplitude, period and phase shift of oscillation.
      SOLN: The amplitude = 1, period = 4 and phase shift = –1/3.

   b. Construct a graph showing the coordinates of at least 5 key points of the function.|

      SOLN: Note that this curve is also described by $x(t) = \sin\left(\dfrac{\pi}{2}t - \dfrac{\pi}{3}\right)$



6. A potter's wheel with radius 4 cm spins at 140 rotations per minute. Find

   a. The angular speed of the wheel in radians per second.

      SOLN: $\dfrac{140\text{rot}}{\text{min}} \times \dfrac{2\pi\text{rad}}{\text{rot}} \times \dfrac{1\text{min}}{60\text{sec}} = \boxed{\dfrac{14\pi\text{rad}}{3\text{sec}} \approx 14.66\text{rad/sec}}$

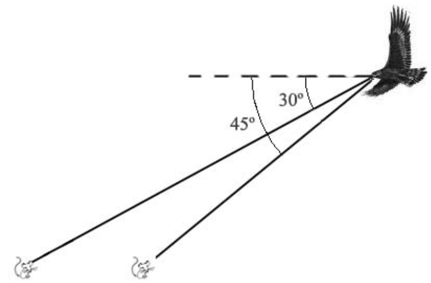
   b. The angular speed of the wheel in degrees per minute.

      SOLN: $\dfrac{140\text{rot}}{\text{min}} \times \dfrac{360^\circ}{\text{rot}} = \boxed{\dfrac{50400^\circ}{\text{min}}}$

   c. The linear speed of a point on the rim of the wheel in meters per second.

      SOLN: $\dfrac{14\pi\text{rad}}{3\text{sec}} \times 4\text{cm} \times \dfrac{1\text{m}}{100\text{cm}} = \dfrac{14\pi m}{75\text{sec}} \approx 0.5864\text{m/s}$

7. A bird observes the angles of depression to two rodents on the ground to be 45º and 30º, as shown. If the bird is flying at an elevation of 100 meters, find the distance between the rodents.



SOLN: Let d_1 = the distance from the farther rodent to the point directly below the bird and let d_2 = the distance from the nearer rodent to the point directly below the bird. Then $\tan 45^\circ = \dfrac{100}{d_2} \Leftrightarrow d_2 = \dfrac{100}{\tan 45^\circ} = 100$

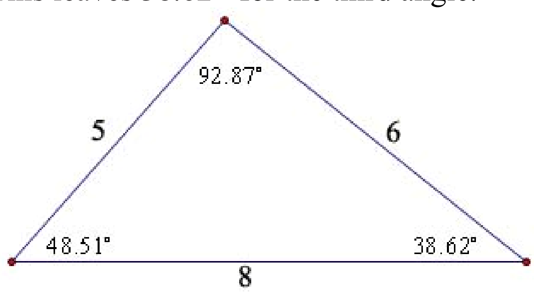
and $\tan 30^\circ = \dfrac{100}{d_1} \Leftrightarrow d_2 = \dfrac{100}{\tan 30^\circ} = 100\sqrt{3}$ so the distance between the

rodents is $d_1 - d_2 = 100\sqrt{3} - 100 = 100\left(\sqrt{3}-1\right) \approx 73.21$ meters

8. Consider the triangle with sides of lengths 5, 6 and 8 meters.
   a. Solve the triangle. That is, find the degree measures of the interior angles.
      SOLN: We need the law of cosines to get one angle: $\theta = \cos^{-1}\left(\dfrac{25+36-64}{60}\right) = \cos^{-1}\left(\dfrac{-1}{20}\right) \approx 92.87^\circ$

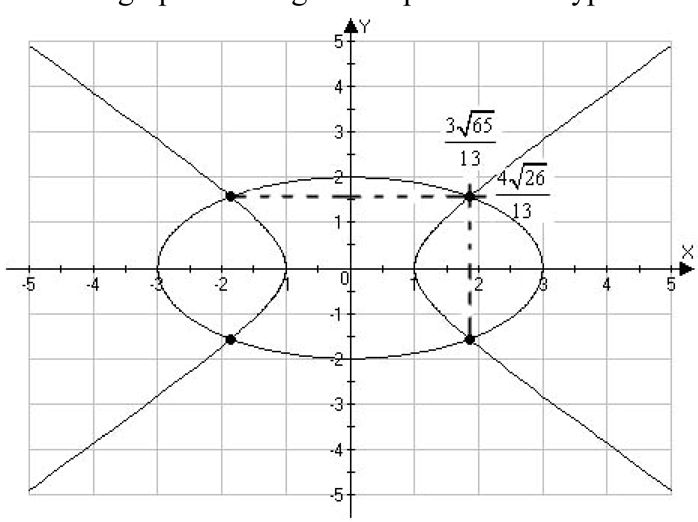
      Then the law of sines to get another: $\dfrac{\sin\theta}{6} \approx \dfrac{\sin 92.87^\circ}{8} \Rightarrow \theta \approx \sin^{-1}\left(\dfrac{3\sin 92.87^\circ}{4}\right) \approx 48.51^\circ$

      This leaves 38.62º for the third angle.



   b. Find the area of the triangle.
      SOLN: The semiperimater is 9.5 so Heron's formula gives √((9.5)(1.5)(3.5)(4.5)) =
      $\sqrt{\dfrac{19\cdot 3\cdot 7\cdot 9}{16}} = \dfrac{3}{4}\sqrt{399} \sim 14.98\ \text{m}^2$.

9. Consider the ellipse described by $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and the hyperbola described by $x^2 - y^2 = 1$
   a. Sketch a graph showing the ellipse and the hyperbola together.



   b. What are the coordinates of the points where the two conic sections intersect?
      SOLN: Substitue $y^2 = x^2 - 1$ from the hyperbola into the equation for the ellipse:

$$\frac{x^2}{9}+\frac{x^2-1}{4}=1 \Leftrightarrow 4x^2+9x^2-9=36 \Leftrightarrow 13x^2=45 \Leftrightarrow \boxed{x=\pm 3\sqrt{\frac{5}{13}}=\pm\frac{3\sqrt{65}}{13}\approx\pm 1.86}$$

Thus $y^2 = x^2 - 1 = 32/13$ so that $y=\pm\sqrt{\frac{32}{13}}=\pm\frac{4\sqrt{26}}{13}\approx\pm 1.57$

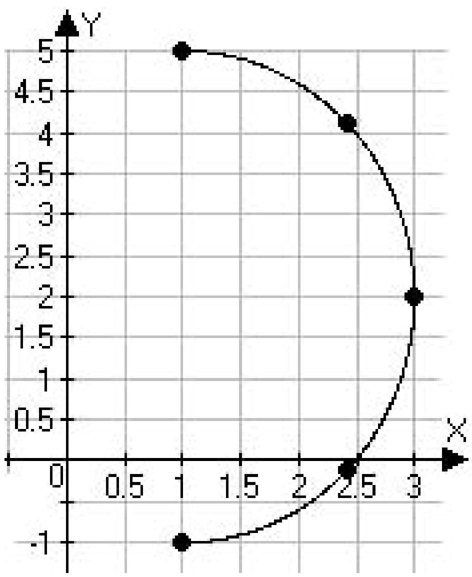
10. Consider the ellipse parameterized by $x(t) = 1+2\sin(t)$ and $y(t) = 2 + 3\cos(t)$

a. Write the standard form for the equation of this ellipse

SOLN: $\left(\frac{x-1}{2}\right)^2+\left(\frac{y-2}{3}\right)^2=1$

a. Complete the table and use the values to sketch a graph for the part of ellipse where $0 \le t \le \pi$

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	1	$1+\sqrt{2}\approx 2.414$	3	$1+\sqrt{2}\approx 2.414$	1
y	5	$2+\frac{3\sqrt{2}}{2}\approx 4.121$	2	$2-\frac{3\sqrt{2}}{2}\approx -0.121$	-1



b.