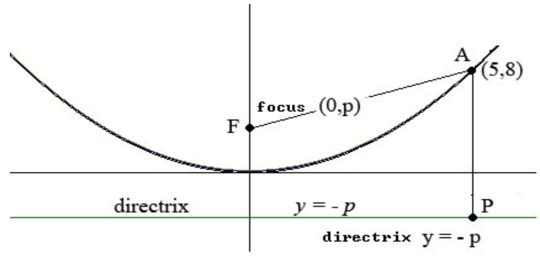
Math 5 – chapter 5 Fair Game – Spring '11

1. Suppose a parabola has its vertex at the origin, opens upward in the xy-plane and passes through the point (5,8).
   a. Write an equation that uses the distance formula to say distance AF = distance AP.
   b. Find the value of p.
   c. What is the length of the focal diameter?



2. Consider the parabola described by $y = 4x^2$.
   a. Write the coordinates of the focus.
   b. Write the equation of the directrix.
   c. Find the focal diameter.
   d. Sketch a graph of the parabola showing these features.

3. Consider the ellipse with foci at (1,0) and (–1,0) and major axis of length $2a = 10$.
   a. Find the length of the minor axis.
   b. Write an equation for the ellipse.
   c. Construct a careful, large graph for the ellipse showing these features.

4. Consider the hyperbola described by $4x^2 - 9y^2 = 36$.
   a. Find the coordinates of the vertices of the hyperbola.
   b. Find the equations for the asymptotes of the hyperbola.
   c. Find the coordinates of the x-intercepts and y-intercepts of the hyperbola.
   d. Construct a careful, large graph the hyperbola, showing all the above features.

5. Write each of the following conic section equations in standard form. Hint: complete squares for x and y and balance equations.
   a. $3x^2 + 2y^2 - 6x - 8y + 5 = 0$
   b. $y^2 - 4x^2 - 2y + 16x = 19$

6. Find the center, vertices, and construct a careful, large graph for the ellipse described by
   $x = 2 + 3\cos t$
   $y = 1 + 5\sin t$

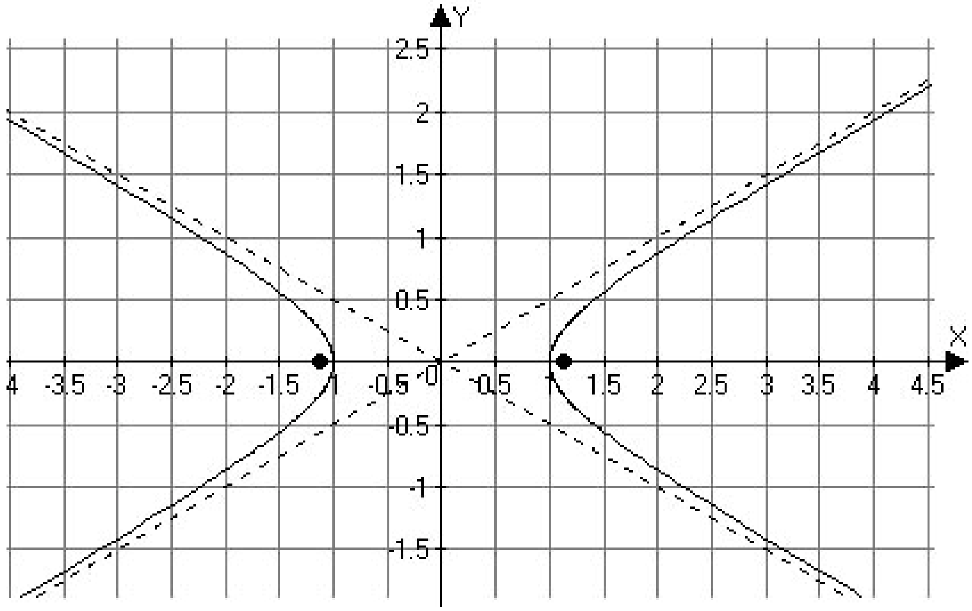
7. Find the center, vertices, asymptotes and construct a careful, large graph for the hyperbola described by
   $x = 2 - \sec t$
   $y = 1 + 2\tan t$

8. Write the standard form for the equation of the hyperbola with asymptotes $y - 2 = \pm(x - 3)$ and foci at (x, y) = (–1,2) and (7,2). Then give parametric equations for this hyperbola.

   Consider the conic section described by $x^2 = 6(x - 2y)$.
   a. Find the coordinates of the vertex.
   b. Find the coordinates of the focus.
   c. Find an equation for the directrix.
   d. Find the focal diameter.
   e. Construct a graph showing these features.

9. Consider the ellipse centered at (12,13) with tangent lines along the coordinate axes.
   a. Where are the vertices? Give coordinates.
   b. Where are the foci? Give coordinates.
   c. Find the eccentricity.
   d. Write parametric equations for this conic.
   e. Construct a careful graph showing the key features.

10. Find an equation for the hyperbola whose graph is shown below:



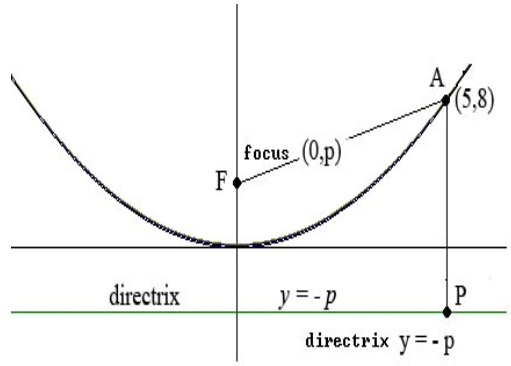
11. Write a polar equation of a conic with the focus at the origin and the given data. Tabulate the *x*-intercepts and *y*-intercepts and sketch a graph for each.
   a. An ellipse with vertex at $(r, \theta) = (3, \pi)$ and eccentricity 0.25

   b. A hyperbola with vertex at $(r, \theta) = (1.2, \pi/2)$ and eccentricity 1.5

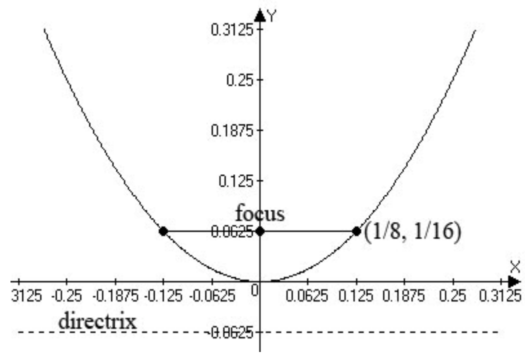
12. Consider the curve given by parametric equations
$$\begin{cases} x = 3\sec(t) \\ y = 5 + 4\tan(t) \\ 0 \le t \le 2\pi \end{cases}$$
   a. Eliminate the parameter *t* to obtain an equation for this curve in rectangular coordinates.
   b. Construct a careful graph for the curve.

Math 5 – Some Solutions – Fair Game '11

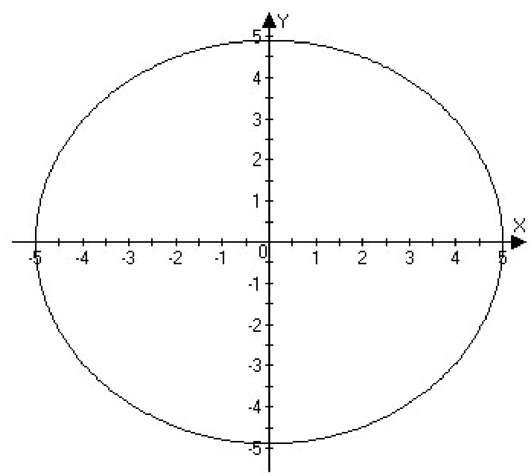
1. Suppose a parabola has its vertex at the origin, opens upward in the *xy*-plane and passes through the point (5,8).
   a. Write an equation that uses the distance formula to say
   distance AF = distance AP.
   SOLN: $\sqrt{25+(8-p)^2} = 8+p$
   b. Find the value of *p*.
   $25+(8-p)^2 = (8+p)^2 \Leftrightarrow 25 = 32p$ so
   $p = 25/32$
   c. What is the length of the focal diameter?
   The length of the focal diameter is $4p = 25/8$.



2. Consider the parabola described by $y = 4x^2$.
   a. Write the coordinates of the focus.
   SOLN: $p = 1/16$ so the focus is at (0,1/16).
   b. Write the equation of the directrix.
   SOLN: The directrix is along $y = -1/16$
   c. Find the focal diameter.
   SOLN: The focal diameter is $4p = -1/4$.
   d. Sketch a graph of the parabola showing these features.
   SOLN: See plot at right.

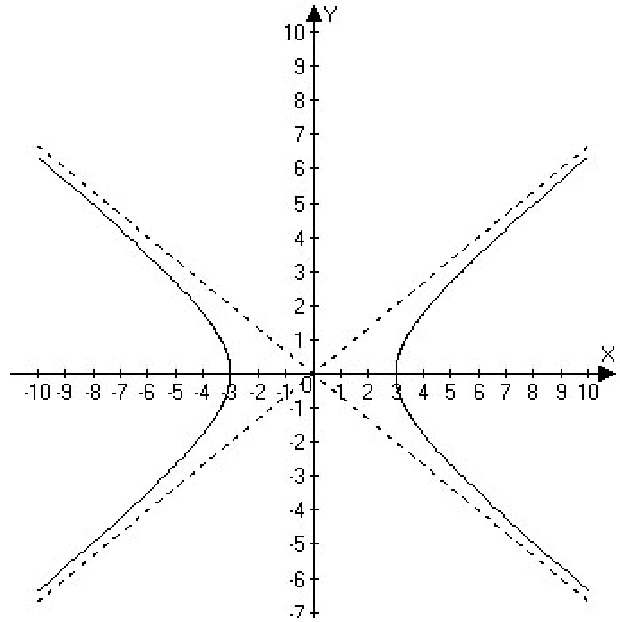


3. Consider the ellipse with foci at (1,0) and (–1,0) and major axis of length $2a = 10$.
   a. Find the length of the minor axis.
   SOLN: Evidently $a = 5$ and $c = 1$, so $b^2 = a^2 - c^2$ means that the length of the minor axis is $2b = 4\sqrt{6}$.
   b. Write an equation for the ellipse.
   SOLN: $\dfrac{x^2}{25}+\dfrac{y^2}{24}=1$
   c. Construct a careful, large graph for the ellipse showing these features.
   SOLN: At right. Note that the eccentricity is 0.2, rather small – so the ellipse is nearly circular.



4. Consider the hyperbola described by $4x^2 - 9y^2 = 36$.
   a. Find the coordinates of the vertices of the hyperbola.

      SOLN: The standard form is $\frac{x^2}{9} - \frac{y^2}{4} = 1$, the vertices are at (–3,0) and (3,0).
   b. Find equations for the asymptotes of the hyperbola.
      SOLN: The asymptotes are $y = \pm 2x/3$
   c. Find the coordinates of the x-intercepts and y-intercepts of the hyperbola.
      SOLN: There are no y-intercepts. The x-intercepts are the vertices at (–3,0) and (3,0).
   d. Construct a careful, large graph the hyperbola, showing all the above features.



5. Write each of the following conic section equations in standard form. Hint: complete squares for x and y and balance equations.
   a. $3x^2 + 2y^2 - 6x - 8y + 5 = 0$

      SOLN: $3\left(x^2 - 2x\right) + 2\left(y^2 - 4y\right) = -5 \Leftrightarrow 3\left(x-1\right)^2 + 2\left(y-2\right)^2 = -5 + 3 + 8$

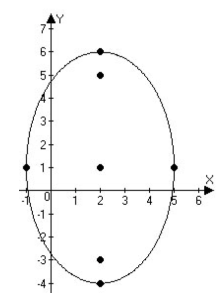
      $\Leftrightarrow \boxed{\frac{\left(x-1\right)^2}{2} + \frac{\left(y-2\right)^2}{3} = 1}$
   b. $y^2 - 4x^2 - 2y + 16x = 19$
      SOLN:
      $y^2 - 4x^2 - 2y + 16x = 19 \Leftrightarrow y^2 - 2y - 4\left(x^2 - 4x\right) = 19 \Leftrightarrow \left(y-1\right)^2 - 4\left(x-2\right)^2 = 19 + 1 - 16$

      $\boxed{\frac{\left(y-1\right)^2}{4} - \left(x-2\right)^2 = 1}$

6. Find the center, vertices, and construct a careful, large graph for the ellipse described by

   $x = 2 + 3\cos t \Leftrightarrow \cos t = \frac{x-2}{3}$

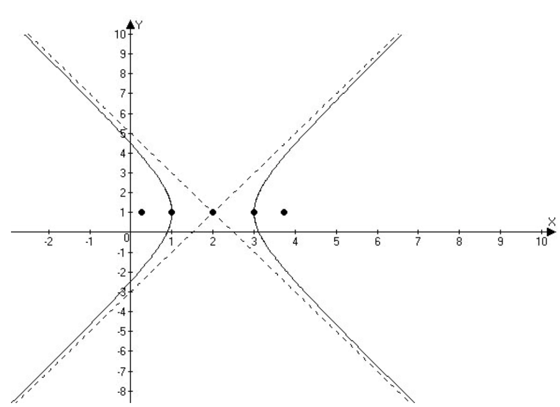
   $y = 1 + 5\sin t \Leftrightarrow \sin t = \frac{y-1}{5}$ So that $\frac{\left(x-2\right)^2}{9} - \frac{\left(y-1\right)^2}{25} = 1$



7. Find the center, vertices, asymptotes and construct a careful, large graph for the hyperbola described by
   $x = 2 - \sec t \Leftrightarrow \sec t = 2 - x$

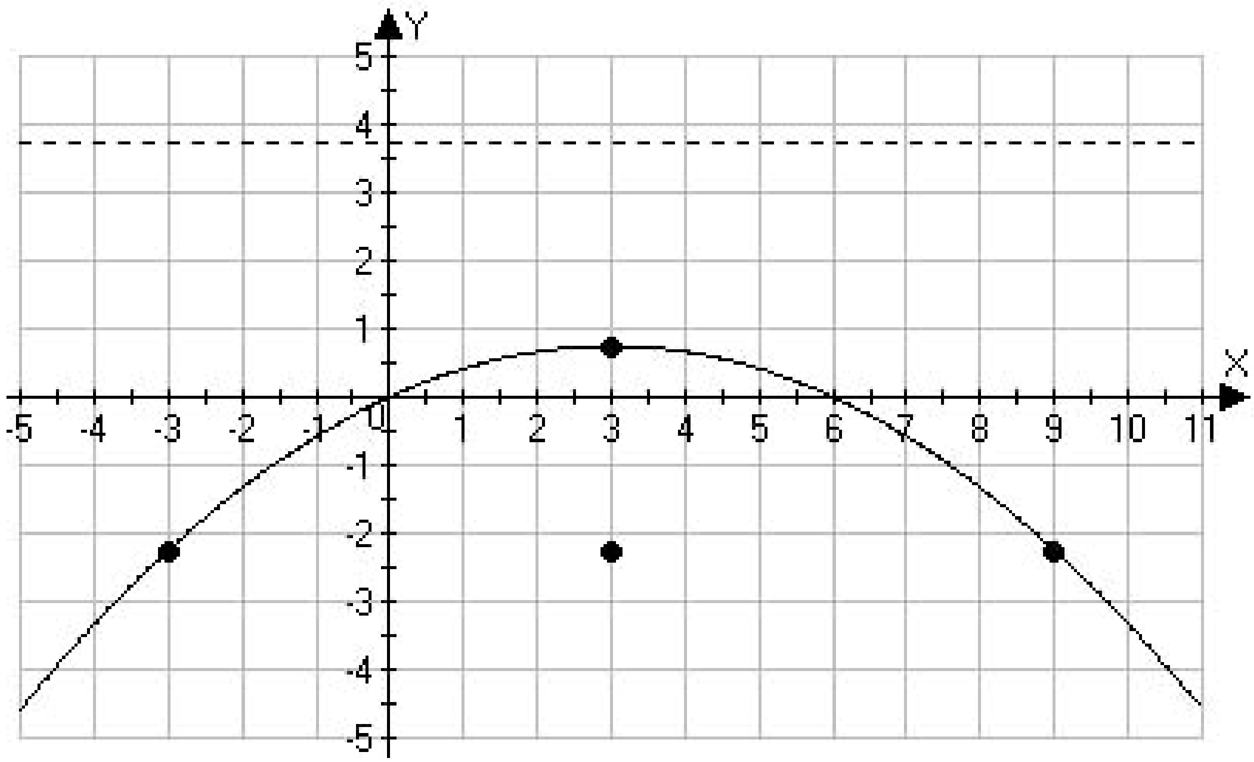
   $y = 1 + 2\tan t \Leftrightarrow \tan t = \frac{y-1}{2}$ so that

   $\left(x-2\right)^2 - \frac{\left(y-1\right)^2}{4} = 1$ with asymptotes $y = 1 + 5(x - 2)/3$ or $y = 1 - 5(x - 2)/3$



8. Write the standard form for the equation of the hyperbola with asymptotes $y-2=\pm(x-3)$ and foci at $(x, y) = (–1,2)$ and $(7,2)$. Then give parametric equations for this hyperbola.

9. Consider the conic section described by $x^2 = 6(x – 2y)$.
   a. Find the coordinates of the vertex.
      SOLN: $x^2 = 6(x-2y) \leftrightarrow x^2 - 6x = -12y \leftrightarrow x^2 - 6x + 9 = -12y + 9 \leftrightarrow (x-3)^2 = -12(y - \frac{3}{4})$
      so the vertex is at $(3, \frac{3}{4})$
   b. Find the coordinates of the focus.
      SOLN: $p = 3$ and the parabola opens downwards from its vertex,
      so the focus is at $\left(3, \frac{3}{4}-3\right) = \left(3, -\frac{9}{4}\right)$
   c. Find an equation for the directrix.
      SOLN: The directrix is a horizontal line 3 units above the vertex: $y = \frac{15}{4}$
   d. Find the focal diameter.
      SOLN: The focal diameter has length 12 and extends from $\left(-3, -\frac{9}{4}\right)$ to $\left(6, -\frac{9}{4}\right)$.
   e. Construct a graph showing these features.
      SOLN:



10. Consider the ellipse centered at (12,13) with tangent lines along the axes, $x = 0$ and $y = 0$.

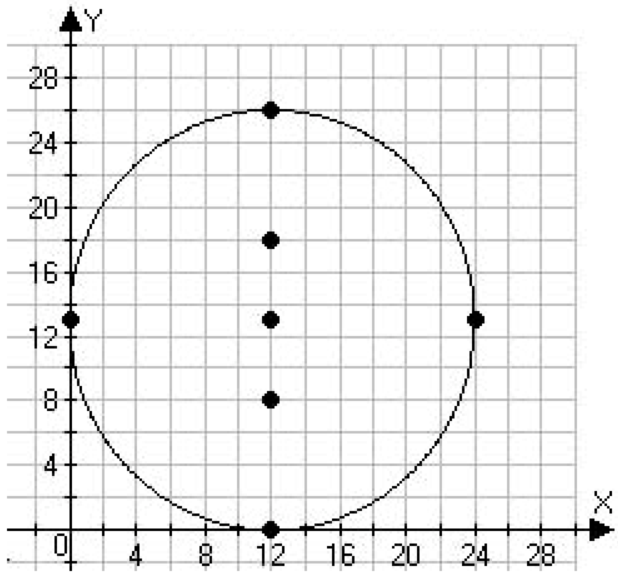
a. Where are the vertices? Give coordinates.
SOLN: (12,0) and (12,26) are the major axis vertices and (0,13) and (24,13) are on the minor axis.

b. Where are the foci? Give coordinates.
SOLN: $c^2 = a^2 - b^2 = 13^2 - 12^2 = 169 - 144 = 25$, so $c = 5$ and foci at $(12,13\pm5) = (12,8)$ and (12 18).

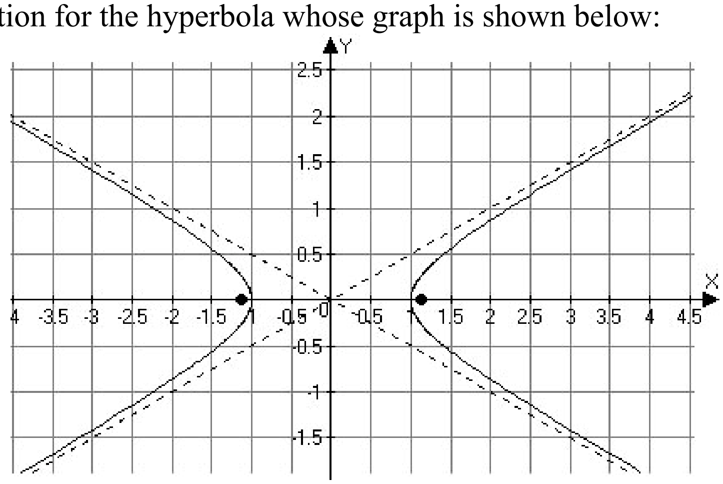
c. Find the eccentricity.
SOLN: eccentricity = $c/a = 5/13$.

d. Write parametric equations for this conic.
SOLN: $x = 12 + 12\cos(t)$ and $y = 13 + 13\sin(t)$.

e. Construct a careful graph showing the key features.



11. Find an equation for the hyperbola whose graph is shown below:



SOLN: Evidently, the center is at (0,0) and the vertices are at (±1,0) and the slopes of the asymptotes are ±½ so $a = 1$ and $b/a = ½$. Thus $b = ½$. Combining this information with the formula $c^2 = a^2 + b^2 = 5/4$

The equation of the for the hyperbola is then $x^2 - 4y^2 = 1$

12. Write a polar equation of a conic with the focus at the origin and the given data.
Tabulate the *x*-intercepts and *y*-intercepts and sketch a graph for each.

a. An ellipse with vertex at $(r, \theta) = (3, \pi)$ and eccentricity 0.25
There are two good solutions to this problem:
SOLN1: With a focus is at (0,0), the ellipse could open to the right from the vertex at $(x, y) = (-3, 0)$ and so the formula would be of the type $r = \dfrac{ed}{1 - e\cos\theta} = \dfrac{d/4}{1 - \dfrac{\cos\theta}{4}} = \dfrac{d}{4 - \cos\theta}$

.

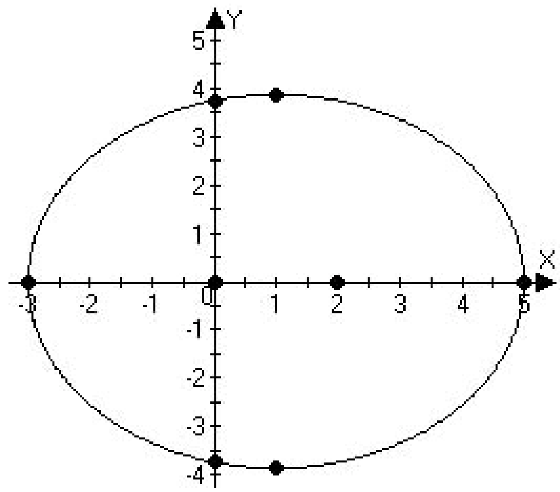
At $\theta = \pi$, $3 = \dfrac{d}{4 - \cos\pi} = \dfrac{d}{5} \Rightarrow d = 15$ so the polar equation for ellipse is $\boxed{r = \dfrac{15}{4 - \cos\theta}}$.

The intercepts are

r	5	15/4	3	15/4
θ	0	$\pi/2$	π	$3\pi/2$

$\Leftrightarrow$

x	5	0	−3	0
y	0	15/4	0	15/4



SOLN2: …or the ellipse could open to the left and still have a vertex at $(x, y) = (-3, 0)$. In this case the formula is of the type $r = \dfrac{ed}{1 + e\cos\theta} = \dfrac{d/4}{1 + \dfrac{\cos\theta}{4}} = \dfrac{d}{4 + \cos\theta}$.

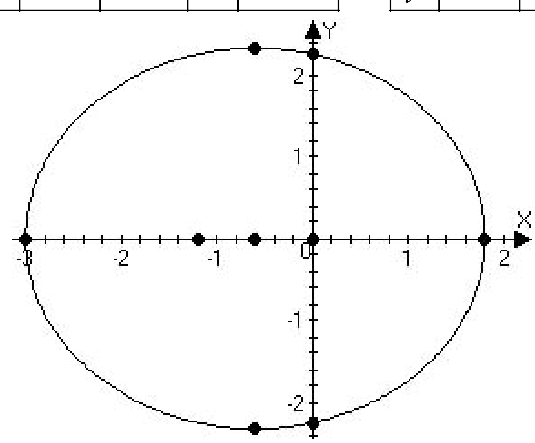
At $\theta = \pi$, $3 = \dfrac{d}{4 + \cos\pi} = \dfrac{d}{3} \Rightarrow d = 9$ so the polar equation for ellipse is $\boxed{r = \dfrac{9}{4 + \cos\theta}}$.

The intercepts are

r	9/5	9/4	3	9/4
θ	0	$\pi/2$	π	$3\pi/2$

$\Leftrightarrow$

x	9/5	0	−3	0
y	0	9/4	0	9/4



b. A hyperbola with vertex at $(r, \theta) = (1.2, \pi/2)$ and eccentricity 1.5

SOLN: Here the vertex is at $(x,y) = (0,6/5)$ and so that branch of the hyperbola opens downward and the formula is of the type $r = \dfrac{ed}{1+e\sin\theta} = \dfrac{3d/2}{1+\dfrac{3\sin\theta}{2}} = \dfrac{3d}{2+3\sin\theta}$.

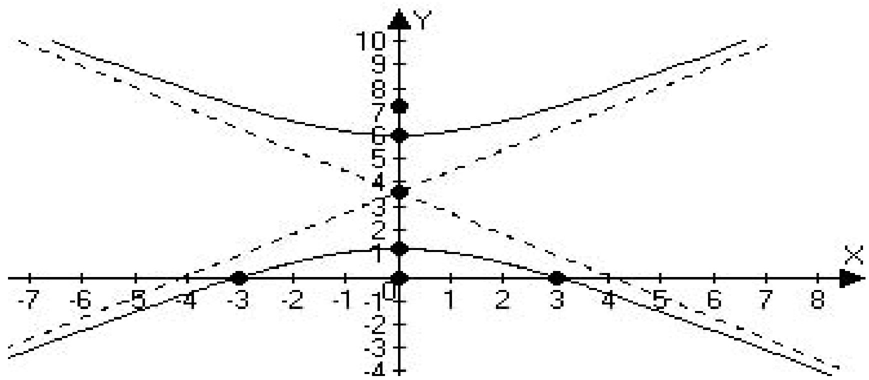
At $\theta = \pi/2$, $\dfrac{6}{5} = \dfrac{3d}{2+3\sin\dfrac{\pi}{2}} = \dfrac{3d}{5} \Rightarrow d = 2$ so the equation is $\boxed{r = \dfrac{6}{2+3\sin\theta}}$.

The intercepts are

r	3	$6/5$	3	-6
θ	0	$\pi/2$	π	$3\pi/2$

$\Leftrightarrow$

x	3	0	-3	0
y	0	$6/5$	0	6



13. Consider the curve given by parametric equations $\boxed{\begin{array}{l} x = 3\sec(t) \\ y = 5 + 4\tan(t) \\ 0 \le t \le 2\pi \end{array}}$

a. Eliminate the parameter t to obtain an equation for this curve in rectangular coordinates.

SOLN: $\dfrac{x^2}{9} - \dfrac{(y-5)^2}{16} = 1$

b. Construct a careful graph for the curve.